The geometry monoid $G_{LD}$ that describes the action of the left self-distributivity identity on terms is not a group, and, contrary to the case of associativity, we cannot obtain a group by merely using a projection. However, we know a family of relations holding in $G_{LD}$, namely these LD-relations that define $\equiv^\oplus$, and we have observed that many properties of $G_{LD}$ can be established by using these relations exclusively. In this chapter, we investigate the abstract group $G_{LD}$ for which LD-relations form a presentation. The hypothesis that $G_{LD}$ must resemble $G_{LD}$ is kept as a leading principle, and indeed we can show that all geometrical parameters defined in $G_{LD}$ admit counterparts in $G_{LD}$. On the other hand, the group $G_{LD}$ turns out to be an extension of the braid group $B_\infty$—this being the precise content of our slogan: “The geometry of left self-distributivity is an extension of the geometry of braids.” Many results about $G_{LD}$ and $B_\infty$ originate in this connection. In particular, braid exponentiation and braid ordering come from an operation and a relation on $G_{LD}$ that somehow explain them and make their construction natural.

Technically, the group $G_{LD}$ behaves like a sort of generalized Artin group. It shares several properties with such groups, yet a number of technical problems arise from $G_{LD}$, contrary to $B_\infty$, not being a direct limit of finite type groups.

The divisions of the chapter are as follows. In Section 1, we introduce the group $G_{LD}$ and the corresponding monoid $M_{LD}$. We observe that the braid group $B_\infty$ is a quotient of $G_{LD}$, a result connected with the action of braids on left self-distributive systems. We also observe that the presentation of $G_{LD}$ is associated with a complement, and verify that this complement satisfies all
conditions of Chapter II. In Section 2, we embed \( T_1 \) into \( G_{LD} \) by using the absorption property of Chapter V to associate with every term a distinguished word that describes its construction. We deduce a complete description of the connection between \( G_{LD} \) and \( G_{LD} \), and explain how braid exponentiation arises. By extending the approach to the case of several variables, we show how charged braids then appear naturally. In Section 3, we introduce two different (pre)-orders on the group \( G_{LD} \). The first is a preorder connected with the braid ordering, and using it gives a purely syntactical proof for the acyclicity property of free LD-systems, one that does not use braid exponentiation. The second relation is a linear ordering on \( G_{LD} \) which is compatible with multiplication on both sides. In Section 4, we show that shifting the addresses gives a family of injective endomorphisms of \( G_{LD} \), which amounts to determining certain parabolic subgroups of \( G_{LD} \). Finally, we introduce in Section 5 the notion of a simple element in the monoid \( M_{LD} \), which is an exact analog of the notion of a simple braid in \( B_{\infty} \). We establish in particular a normal form result in \( M_{LD} \) which directly extends the braid normal form of Chapter II.

1. The Group \( G_{LD} \) and the Monoid \( M_{LD} \)

Here we consider the group \( G_{LD} \) and the monoid \( M_{LD} \) for which the LD-relations of Chapter VII form a presentation. In this section, we investigate the connection between \( G_{LD} \) and Artin’s braid group \( B_{\infty} \), we observe that the presentation of \( G_{LD} \) and \( M_{LD} \) is associated with a complement as defined in Chapter II, and we establish that this complement is coherent and convergent, which allows us to deduce several significant properties of \( M_{LD} \) and \( G_{LD} \).

**LD-relations vs. braid relations**

We recall that \( A \) denotes the set of all binary addresses, \( i.e., \) of all finite sequences of 0’s and 1’s; \( A^* \) denotes the free monoid generated by \( A \), \( i.e., \) the set of all words on \( A \), while \((A \cup A^{-1})^* \) denotes the free monoid generated by \( A \) and a disjoint copy \( A^{-1} \) of \( A \). The words on \( A \) are said to be positive.

**Definition 1.1.** (group \( G_{LD} \)) We define \( G_{LD} \) to be the group generated by a family of elements \( \{g_\alpha ; \alpha \in A\} \) subject to the relations

\[
\begin{align*}
g_\alpha g_\beta &= g_\beta g_\alpha \quad \text{for} \ \alpha \bot \beta \quad \text{(type \( \bot \))} \\
g_{\alpha \beta \beta} g_\alpha &= g_\alpha g_{\alpha 10 \beta} g_{\alpha 00 \beta} \quad \text{(type 0)} \\
g_{\alpha 10 \beta} g_\alpha &= g_\alpha g_{\alpha 01 \beta} \quad \text{(type 10)}
\end{align*}
\]
SEC. VIII.1: THE GROUP $G_{LD}$ AND THE MONOID $M_{LD}$

\[ g_{α1β}g_α = g_α g_{α1β} \quad \text{(type 11)} \]
\[ g_α g_α g_{α0} = g_α g_{α1} g_α \quad \text{(type 1)} \]

The submonoid of $G_{LD}$ generated by all $g_α$’s is denoted $G_{LD}^+$. The relations that define $G_{LD}$ are the LD-relations of Section VII.2. Hence, by construction, the mapping $α \mapsto g_α$ induces an isomorphism of $(A \cup A^{-1})^+ / \equiv$ onto $G_{LD}$, where $\equiv$ is the congruence considered in Chapter VII.

**Proposition 1.2. (projection)** The mapping $pr$ defined by

\[ pr(g_α) = \begin{cases} 1 & \text{if the address } α \text{ contains at least one 0}, \\ σ_{i+1} & \text{for } α = 1^i, \ i.e., \ 11\ldots1, \ i \text{ times} \end{cases} \]

induces a surjective homomorphism of $G_{LD}$ onto $B_∞$.

**Proof.** Let $f$ be the homomorphism of the free monoid $(A \cup A^{-1})^+$ into $B_∞$ that maps $α$ to $pr(g_α)$. Then $f(u) = f(u')$ holds for every LD-relation $(u, u')$. Let us consider for instance the LD-relation $(α1 · α1 · α1 · α0, α · α1 · α)$. If $α$ contains one 0 at least, both sides are collapsed. For $α = 1^i$, they are mapped to $σ_{i+2}σ_{i+1}σ_{i+2}$ and $σ_{i+1}σ_{i+2}σ_{i+1}$ respectively, hence equal in $B_∞$. ■

By definition, the kernel of $pr$ is the normal subgroup of $G_{LD}$ generated by those elements $g_α$ where $α$ contains at least one 0. The subgroup $\text{Ker}(pr)$ is very large. No complete description is known, but we shall see in Section 4 that it includes in particular infinitely many copies of $G_{LD}$ itself.

The existence of the projection of $G_{LD}$ onto $B_∞$ is connected with the action of braids on LD-systems considered in Chapters I and III. Assume that $S$ is a binary system. Then $n$-strand braid words act on $S^n$ by

\[ σ_1 : (a_1, a_2, a_3, \ldots, a_n) \longmapsto (a_1 a_2, a_1, a_3, \ldots, a_n). \]

If we identify the finite sequence $(a_1, \ldots, a_n)$ with the binary tree

![Binary tree](image)

the action becomes:

\[ σ_1 : \quad \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \ldots \\ \end{array} \mapsto \begin{array}{c} a_1 a_2 \\ a_1 \\ a_3 \\ \ldots \\ \end{array} \]

Now, the self-distributivity operators $LD_α$ also act on binary trees. For instance, we have

\[ LD_β : \quad \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \ldots \\ \end{array} \mapsto \begin{array}{c} a_1 a_2 a_1 \\ a_1 \\ a_3 \\ \ldots \\ \end{array} \]
The only difference between the two actions is that, in the one case, we evaluate the expression \( a_1a_2 \) in \( S \), while, in the other case, we keep it as a formal product. Now, let us consider the action of the equivalent braid words \( \sigma_1\sigma_2\sigma_1 \) and \( \sigma_2\sigma_1\sigma_2 \), and compare it with the action of the corresponding operators \( LD_{\phi_1\phi} \) and \( LD_{\phi_1\phi} \). As we see in Figure 1.1, in the case of braids, the actions of \( \sigma_1\sigma_2\sigma_1 \) and \( \sigma_2\sigma_1\sigma_2 \) coincide provided the operation on \( S \) is left self-distributive. In the case of self-distributivity operators, the actions of \( LD_{\phi_1\phi} \) and \( LD_{\phi_1\phi} \) do not coincide, but introducing the operator \( LD_0 \) is what is needed to obtain the equality. So \( g_0 \) measures the obstruction to an action of braids when the system \( S \) does not satisfy Identity \( (LD) \). Now, when \( S \) is an LD-system, we can collapse the additional term \( g_0 \), and what remains is the action of braids. So, the action of \( B_\infty \) is what remains from the action of \( G_{LD} \) when we collapse all operators but those acting on the rightmost branch of the tree.

**The monoid \( M_{LD} \)**

As the positive geometry monoid \( G_{LD}^+ \) plays a significant role beside \( G_{LD} \), it is natural to introduce the monoid for which LD-relations form a presentation.

**Definition 1.3. (monoid \( M_{LD} \))** We define \( M_{LD} \) to be the monoid with the same presentation as \( G_{LD} \); we use \( g_0^+ \) for the generator associated with \( \alpha \).

By construction, the mapping \( \alpha \mapsto g_0^+ \) induces an isomorphism of \( \mathbb{A}^\times \) onto \( M_{LD} \), and the mapping \( g_0^+ \mapsto g_0 \) induces a surjective homomorphism of \( M_{LD} \) onto \( G_{LD}^+ \); we shall discuss in Section IX.5 the possible injectivity of this homomorphism, i.e., the question of whether \( M_{LD} \) embeds in \( G_{LD} \).

As in the case of \( G_{LD} \), mapping \( g_0^+ \) to 1 if the address \( \alpha \) contains at least one 0, and to \( \sigma_{\alpha+1} \) for \( \alpha = 1^i \) defines a surjective homomorphism of \( M_{LD} \) onto \( B_{\infty}^+ \).

For each LD-relation \( (u, u') \), and for each address \( \gamma \), the shifted pair \( (\gamma u, \gamma u') \) is still an LD-relation. We easily obtain:
**Proposition 1.4.** (shift) For each address $\gamma$, the mapping $sh_\gamma : g^\alpha \mapsto g^\gamma g^\alpha$ induces an endomorphism of $G_{LD}$, and the mapping $sh^+_\gamma : g^+ \mapsto g^+ g^\gamma$ induces an injective endomorphism of $M_{LD}$.

**Proof.** The existence follows from the invariance of LD-relations under shift. To prove injectivity in the case of $M_{LD}$, we observe that, if $(u,u')$ is a LD-relation and all generators occurring in $u$ belong to the image of $sh_\gamma$, so do all generators occurring in $u'$. Hence, if $\gamma v \equiv \gamma v'$ holds, all intermediate words in a sequence of elementary transformations from $\gamma v$ to $\gamma v'$ belong to the image of $sh_\gamma$, hence $v \equiv v'$ holds as well. $\blacksquare$

We shall prove in Section 4 that the endomorphisms $sh_\gamma$ of $G_{LD}$ are injective as well, but the proof requires some care, and the result is not needed now.

The crucial point for our study of $M_{LD}$ is that, for every pair of addresses $(\alpha,\beta)$, there exists exactly one LD-relation of the form $g^\alpha \cdots = g^\beta \cdots$. With the definitions of Chapter II, this means that our presentation of $M_{LD}$ is associated with a complement on the right.

**Definition 1.5.** (complement) We denote by $f$ the mapping of $A \times A$ into $A^*$ defined by

$$f(\alpha,\beta) = \begin{cases} 
\alpha 00 \gamma \cdot \alpha & \text{for } \beta = \alpha 00, \\
\alpha 10 \gamma & \text{for } \beta = \alpha 10, \\
\beta \cdot \alpha & \text{for } \beta = \alpha 1, \\
\epsilon & \text{for } \beta = \alpha, \\
\beta \cdot \alpha \cdot \beta & \text{for } \alpha = \beta 1, \\
\beta & \text{for } \alpha \not\subseteq \beta 1 \text{ or } \alpha 11 \subseteq \beta.
\end{cases}$$

By definition, the monoid $M_{LD}$ is the monoid associated with $f$ on the right, while $G_{LD}$ is the corresponding group. As in Chapter II, we shall use $\cap$, $\setminus$, $N$, $R$ for the (right) reversing relation, the extension to positive words, the numerator and the denominator associated with $f$ respectively.

We have observed that the braid group $B_\infty$ is a projection of the group $G_{LD}$, and, similarly, the braid monoid $B^+_{\infty}$ is a projection of the monoid $M_{LD}$. The existence of such surjective morphisms originates in the existence of a similar surjective map for the complements. We denote by $pr$ the surjective homomorphism of $(A \cup A^{-1})^*$ onto $BW_\infty$ that maps $\alpha$ to $\sigma_{i+1}$ if the address $\alpha$ has the form $1^i$, and to $\epsilon$ otherwise. This mapping induces the surjective homomorphism of $G_{LD}$ onto $B_\infty$ considered above, and there is no ambiguity in our notation.

**Lemma 1.6.** (i) The projection $pr$ of $(A \cup A^{-1})^*$ onto $BW_\infty$ is a morphism with respect to the complements $f$ and $f_n$, in the sense that

$$pr(f(\alpha,\beta)) = pr(\alpha) \setminus pr(\beta)$$

(1.1)
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holds for all addresses $\alpha, \beta$.

(ii) If $u, v$ are words on $A$ and $u \setminus v$ exists, we have

$$\text{pr}(u \setminus v) = \text{pr}(u) \setminus \text{pr}(v).$$  \hfill (1.2)

(iii) If $w$ is a word on $A \cup A^{-1}$ and $w \leadsto w'$ holds, i.e., $w$ is reversible to $w'$, then $\text{pr}(w) \leadsto \text{pr}(w')$ holds as well. In particular, the equalities

$$\text{pr}(N(w)) = N_{\text{pr}}(w) \quad \text{and} \quad \text{pr}(D(w)) = D_{\text{pr}}(w)$$  \hfill (1.3)

hold whenever $N(w)$ and $D(w)$ exist.

Proof. Formula (1.1) follows from the definitions. The other results follow from an induction on the number of reversing steps (cf. Exercise II.2.21).

Another similarity between the complement for $G_{LD}$ and the braid complement involves the partial actions on terms and on sequences from an LD-system previously defined. Lemma III.1.4 tells us that, if $w$ is a braid word such that $\vec{a} \cdot w$ is defined, and $w$ is right reversible to $w'$, then $\vec{a} \cdot w'$ is defined as well, and we have $\vec{a} \cdot w' = \vec{a} \cdot w$. A similar phenomenon happens with the complement $f$ and the partial action of $(A \cup A^{-1})^*$ on terms.

**Lemma 1.7.** Assume that $w, w'$ are words on $A \cup A^{-1}$ and $w$ is reversible to $w'$. Assume that $t$ is a term, and $t \cdot w$ exists. Then $t \cdot w'$ exists as well, and we have $t \cdot w' = t \cdot w$.

Proof. It suffices to consider one step of reversing. Let us look at $w = 1^{-1} \cdot \phi$, $w' = \phi \cdot 1 \cdot 0 \cdot \phi^{-1} \cdot 1^{-1}$, actually the most complicated case. Assuming that $t$ lies in the domain of $LD_{1}^{-1} \cdot LD_{\phi}$ implies that $t$ has the form $\begin{array}{c} \text{t1} \\
\text{t2} \\
\text{t3} \\
\text{t4} \end{array}$. Then, $t$

lies in the domain of $LD_{\phi} \cdot LD_{1} \cdot LD_{0}$, and we find $t \cdot \phi \cdot 1 \cdot 0 = \begin{array}{c} \text{t1} \\
\text{t2} \\
\text{t3} \\
\text{t4} \end{array}$, which belongs to the image of $LD_{1} \cdot LD_{\phi}$. Finally, we obtain $t \cdot w = t \cdot w' = \begin{array}{c} \text{t1} \\
\text{t2} \\
\text{t3} \\
\text{t4} \end{array}$.
Section VIII.1: The Group $G_{LD}$ and the Monoid $M_{LD}$

Completeness of word reversing

According to the scheme of Chapter II, we shall study the possible coherence and convergence of the complement $f$ the monoid $M_{LD}$ is associated with. The projection results of Lemma 1.6 are useless here: actually, deduction goes the other direction: proving results about $f$, like coherence or convergence, as will be done below, will re-prove the corresponding results about the braid complement.

Let us begin with norm. Contrary to the case of braids, the length of the words is not preserved under LD-relations, so the problem is nontrivial.

**Proposition 1.8. (atomicity)** The complement $f$ admits a norm, hence the monoid $M_{LD}$ is atomic.

**Proof.** Let us define, for $u$ a word on $A$,

$$\nu(u) = \text{size}(t_{\alpha}^u) - \text{size}(t_{\beta}^u). \quad (1.4)$$

Firstly, $u' \equiv^+ u$ implies $LD_{u} = LD_{u'}$, hence $t_{u}^u = t_{u'}^u$ and $t_{\alpha}^u = t_{\alpha}^{u'}$, and, finally, $\nu(u') = \nu(u)$. Assume that $\alpha$ is an address, and $u$ is a word on $A$. By definition, we have $t_{\alpha}^{u'} = (t_{\alpha}^u \cdot \alpha) \cdot u$, hence there exists a substitution $f$ satisfying $t_{\alpha}^{u'} \cdot \alpha = (t_{\alpha}^uf)$ and $t_{\alpha}^{u'} = (t_{\alpha}^{u})f$. We deduce

$$\nu(\alpha \cdot u) = \text{size}(t_{\alpha}^{u'}) - \text{size}(t_{\alpha}^{u})$$

$$= \text{size}(t_{\alpha}^{u}) - \text{size}(t_{\alpha}^{u'} \cdot \alpha) + \text{size}(t_{\alpha}^{u'} \cdot \alpha) - \text{size}(t_{\alpha}^{u})$$

$$= \text{size}((t_{\alpha}^{u})f) - \text{size}((t_{\alpha}^{u}')f) + \text{size}(t_{\alpha}^{u'} \cdot \alpha) - \text{size}(t_{\alpha}^{u})$$

$$> \text{size}((t_{\alpha}^{u})f) - \text{size}((t_{\alpha}^{u}')f) \geq \text{size}(t_{\alpha}^{u}) - \text{size}(t_{\alpha}^{u}) = \nu(u).$$

A similar argument gives $\nu(u \cdot \alpha) > \nu(u)$, and $\nu$ is a norm for $f$. \hfill \square

It follows that the atoms of the monoid $M_{LD}$ are the elements $g_{\alpha}^+$ and, that, for every word $u$ on $A$, the length of every word $u'$ satisfying $u' \equiv^+ u$ is bounded above by the integer $\nu(u)$ defined in (1.4).

Let us turn to the completeness of word reversing. Owing to Propositions II.2.5 (complete) and II.2.9 (locally coherent), we are left with the question of establishing that $f$ is coherent on $A$.

**Proposition 1.9. (coherence)** The complement $f$ is coherent on $A$.

**Proof.** We consider all triples $(\alpha, \beta, \gamma)$ in $A^3$ and, for each of them, we prove the three relations

$$(\alpha \setminus \beta) \setminus (\alpha \setminus \gamma) \equiv^+ (\beta \setminus \alpha) \setminus (\beta \setminus \gamma),$$

$$(\beta \setminus \gamma) \setminus (\beta \setminus \alpha) \equiv^+ (\gamma \setminus \beta) \setminus (\gamma \setminus \alpha),$$

$$(\gamma \setminus \alpha) \setminus (\gamma \setminus \beta) \equiv^+ (\alpha \setminus \gamma) \setminus (\alpha \setminus \beta),$$

where $\equiv^+$ denotes the relation of equivalence defined by $u' \equiv^+ u$ if and only if $u$ and $u'$ are equivalent under some substitution $f$.\hfill \square
where we recall \( u' \equiv t^+ t \) means that \( u^{-1} u' \) is reversible to \( \varepsilon \). As LD-relations are closed under shift, we may assume that the greatest common prefix of \( \alpha \), \( \beta \), \( \gamma \) is empty, and, because we consider the three cyclic permutations of the addresses simultaneously, we may choose the ordering of \( \alpha, \beta, \gamma \).

**Case 1.** Two addresses are equal. Assume \( \alpha = \beta \). Then we have
\[
(\alpha \setminus \beta) \setminus (\alpha \setminus \gamma) = \varepsilon \setminus (\alpha \setminus \gamma) = \alpha \setminus \gamma,
\]
\[
(\beta \setminus \alpha) \setminus (\beta \setminus \gamma) = \varepsilon \setminus (\beta \setminus \gamma) = \beta \setminus \alpha;
\]
\[
(\beta \setminus \gamma) \setminus (\beta \setminus \alpha) = (\beta \setminus \gamma) \varepsilon = \varepsilon,
\]
\[
(\gamma \setminus \beta) \setminus (\gamma \setminus \alpha) = (\gamma \setminus \beta) (\gamma \setminus \alpha) = \varepsilon,
\]
and this is enough, as \( \alpha \) and \( \beta \) play symmetric roles.

**Case 2.** One address is orthogonal to the greatest common prefix of the two others. We assume that \( \gamma \) is orthogonal to the greatest common prefix of \( \alpha \) and \( \beta \). Then \( \gamma \) is orthogonal to every address occurring in \( \alpha \setminus \beta \) and \( \beta \setminus \alpha \), and we have
\[
(\alpha \setminus \beta) \setminus (\alpha \setminus \gamma) = (\alpha \setminus \beta) \setminus \gamma = \gamma,
\]
\[
(\beta \setminus \alpha) \setminus (\beta \setminus \gamma) = (\beta \setminus \alpha) \setminus \gamma = \gamma;
\]
\[
(\beta \setminus \gamma) \setminus (\beta \setminus \alpha) = (\gamma \setminus (\beta \setminus \alpha)) = \beta \setminus \alpha,
\]
\[
(\gamma \setminus \beta) \setminus (\gamma \setminus \alpha) = \beta \setminus \alpha.
\]
Again this is enough, since \( \alpha \) and \( \beta \) play symmetric roles.

**Case 3.** One address is a prefix of the greatest common prefix of the two others. We assume that \( \gamma \) is a prefix of the greatest common prefix \( \delta \) of \( \alpha \) and \( \beta \). As we assumed that \( \alpha, \beta \) and \( \gamma \) have no common prefix, we have \( \gamma \neq \varepsilon \).

**Case 3.1.** The address 0 is a prefix of \( \alpha \) and \( \beta \). Write \( \alpha = \alpha_0, \beta = \beta_0 \), and assume \( \alpha_0 \setminus \beta_0 = \beta_1 \cdots \beta_p \) and \( \beta_0 \setminus \alpha_0 = \alpha_1 \cdots \alpha_p \). Then we have
\[
(\alpha \setminus \beta) \setminus (\alpha \setminus \gamma) = (\alpha_0 \setminus \beta_0) \setminus (\beta_0 \setminus \alpha_0) = \phi,
\]
\[
(\beta \setminus \alpha) \setminus (\beta \setminus \gamma) = (\alpha_0 \setminus \beta_0) \setminus (\beta_0 \setminus \alpha_0) = \phi;
\]
\[
(\beta \setminus \gamma) \setminus (\beta \setminus \alpha) = (\beta_0 \setminus \alpha_0) = 10 \alpha_1 \cdots 0 \alpha_p,
\]
\[
(\gamma \setminus \beta) \setminus (\gamma \setminus \alpha) = (\beta_0 \setminus \alpha_0) = 10 \alpha_1 \cdots 0 \alpha_p.
\]
It remains to compute
\[
(10 \alpha_1 \cdots 0 \alpha_1 \cdots 0 \alpha_p, 0 \alpha_p) \setminus (10 \alpha_1 \cdots 0 \alpha_p, 0 \alpha_p),
\]
and the symmetric expression. Using type \( \sqsubseteq \) relations between 10a1 and 00a1, we see that the previous words are empty, and we are done.

**Case 3.2.** The address 1 is a proper prefix of \( \alpha \) and \( \beta \). Write \( \alpha = 1 \epsilon \alpha_0, \beta = 1 \epsilon \beta_0 \), with \( \epsilon = 0 \) or \( \epsilon = 1 \). As every factor in \( \alpha \setminus \beta \) and \( \beta \setminus \alpha \) begins with \( 1 \epsilon \), and, for every \( \delta \), we have \( 1 \epsilon \delta \setminus 1 \epsilon \delta = 1 \epsilon \delta \), we obtain
\[
(\alpha \setminus \beta) \setminus (\alpha \setminus \gamma) = (\alpha \setminus \beta) \setminus (\beta \setminus \gamma) = \phi,
\]
\[
(\beta \setminus \alpha) \setminus (\beta \setminus \gamma) = (\beta \setminus \alpha) \setminus (\beta \setminus \gamma) = \phi;
\]
\[
(\beta \setminus \gamma) \setminus (\beta \setminus \alpha) = 1 \epsilon \beta_0 \setminus (\beta_0 \setminus \alpha_0),
\]
\[
(\gamma \setminus \beta) \setminus (\gamma \setminus \alpha) = 1 \epsilon \beta_0 \setminus (\beta_0 \setminus \alpha_0),
\]
which is enough for this case.
Case 3.3.  The address 1 is the greatest common prefix of \( \alpha \) and \( \beta \).

Case 3.3.1.  The addresses \( \alpha \) and \( \beta \) are orthogonal. We may assume \( \alpha = 10\alpha_0 \) and \( \beta = 11\beta_0 \). Then we have
\[
(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma) = 11\beta_1\backslash\phi = \phi,
(\beta\alpha)\backslash(\beta\gamma) = 10\alpha_0\backslash\phi = \phi;
(\beta\gamma)\backslash(\beta\alpha) = \phi\cdot10\alpha_0 = 01\alpha_0,
(\gamma\backslash\beta)\backslash(\gamma\alpha) = 11\beta_1\backslash01\alpha_0 = 01\alpha_0;
(\gamma\alpha)\backslash(\gamma\beta) = 10\alpha_0\backslash11\beta_1 = 11\beta_1,
(\alpha\gamma)\backslash(\alpha\beta) = \phi\cdot11\beta_1 = 11\beta_1,
\]
and we are done.

Case 3.3.2.  The addresses \( \alpha \) and \( \beta \) are comparable. We may assume that \( \beta \) is a strict prefix of \( \alpha \), hence \( \beta = 1 \).

Case 3.3.2.1.  The address 10 is a prefix of \( \alpha \), say \( \alpha = 10\alpha_0 \). We have
\[
(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma) = 1\phi = \phi \cdot 1 \cdot 0,
(\beta\alpha)\backslash(\beta\gamma) = (110\alpha_0 \cdot 100\alpha_0) \backslash (\phi \cdot 1 \cdot 0) = \phi \cdot 1 \cdot 0;
(\beta\gamma)\backslash(\beta\alpha) = (\phi \cdot 1 \cdot 0) \backslash (110\alpha_0 \cdot 100\alpha_0)
= (1 \cdot 0) \backslash (110\alpha_0 \cdot 010\alpha_0) = 0 \backslash (101\alpha_0 \cdot 010\alpha_0) = 101\alpha_0 \cdot 001\alpha_0,
(\gamma\backslash\beta)\backslash(\gamma\alpha) = (1 \cdot \phi) \backslash 01\alpha_0 = \phi \cdot 01\alpha_0 = 101\alpha_0 \cdot 001\alpha_0;
(\gamma\alpha)\backslash(\gamma\beta) = 01\alpha_0 \backslash (1 \cdot \phi) = 1 \cdot \phi,
(\alpha\gamma)\backslash(\alpha\beta) = \phi \cdot 1 = 1 \cdot \phi,
\]
and we are done for this case.

Case 3.3.2.2.  The address 11 is a strict prefix of \( \alpha \). Write \( \alpha = 11e\alpha_0 \) with \( e = 0 \) or \( e = 1 \). We have
\[
(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma) = 1\phi = \phi \cdot 1 \cdot 0,
(\beta\alpha)\backslash(\beta\gamma) = 1e1\alpha_0 \backslash (\phi \cdot 1 \cdot 0) = \phi \cdot 1 \cdot 0;
(\beta\gamma)\backslash(\beta\alpha) = (\phi \cdot 1 \cdot 0) \backslash 1e1\alpha_0 = (1 \cdot 0) \backslash e11\alpha_0 = e11\alpha_0,
(\gamma\backslash\beta)\backslash(\gamma\alpha) = (1 \cdot \phi) \backslash 11\alpha_0 = \phi \cdot 11\alpha_0 = e11\alpha_0;
(\gamma\alpha)\backslash(\gamma\beta) = 11e\alpha_0 \backslash (1 \cdot \phi) = 1 \cdot \phi,
(\alpha\gamma)\backslash(\alpha\beta) = \phi \cdot 1 = 1 \cdot \phi,
\]
and we are done.

Case 3.3.2.3.  The address \( \alpha \) is 11. This case is critical. We find (see Figure 2.1)
\[
(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma) = (11 \cdot 11 \cdot 10) \backslash \phi = \phi \cdot 1 \cdot 11 \cdot 10 \cdot 0 \cdot 01 \cdot 00,
(\beta\alpha)\backslash(\beta\gamma) = (11 \cdot 1) \backslash (\phi \cdot 1 \cdot 0) = \phi \cdot 1 \cdot 0 \cdot 11 \cdot 01 \cdot 10 \cdot 00,
\]
and we then verify \( \phi \cdot 1 \cdot 11 \cdot 10 \cdot 0 \cdot 01 \cdot 00 \equiv 11 \cdot 1 \cdot 0 \cdot 11 \cdot 01 \cdot 10 \cdot 00 \).

Similarly, we have
\[
(\beta\gamma)\backslash(\beta\alpha) = (\phi \cdot 1 \cdot 0) \backslash (11 \cdot 1) = 11 \cdot 1 \cdot \phi,
(\gamma\backslash\beta)\backslash(\gamma\alpha) = (1 \cdot \phi) \backslash 11 = 11 \cdot 1 \cdot \phi,
\]
and we have equality. Finally, we have
\[
(\gamma\alpha)\backslash(\gamma\beta) = 11 \backslash (1 \cdot \phi) = 1 \cdot 11 \cdot 10 \cdot \phi \cdot 1 \cdot 0,
(\alpha\gamma)\backslash(\alpha\beta) = \phi \cdot 1 \cdot 11 \cdot 10 \cdot 11 \cdot 01 \cdot 00 = 11 \cdot 1 \cdot 1 \cdot \phi \cdot 1 \cdot 11 \cdot 01 \cdot 00,
\]
and we verify \( 1 \cdot 11 \cdot 10 \cdot \phi \cdot 1 \cdot 0 \equiv 11 \cdot 1 \cdot 1 \cdot \phi \cdot 1 \cdot 11 \cdot 01 \cdot 00 \).
Case 3.4. The address \( \alpha \) is the greatest common prefix of \( \alpha \) and \( \beta \). We may assume \( \alpha \perp \beta \), for, otherwise, one of \( \alpha, \beta \) should be \( \phi \), and we are in case 1. So we assume \( \alpha = 0\alpha_0 \), while 1 is a prefix of \( \beta \).

Case 3.4.1. The address 1 is a strict prefix of \( \beta \). Write \( \beta = 1e\beta_0 \), with \( e = 0 \) or \( e = 1 \). We find 
\[
(\alpha \backslash \beta) \setminus (\alpha \backslash \gamma) = 1e\beta_0 \backslash \phi = \phi, \\
(\beta \backslash \alpha) \setminus (\beta \backslash \gamma) = 0\alpha_0 \backslash \phi = \phi, \\
(\beta \backslash \gamma) \setminus (\beta \backslash \alpha) = \phi_0\alpha_0 = 10\alpha_0 \cdot 00\alpha_0, \\
(\gamma \backslash \beta) \setminus (\gamma \backslash \alpha) = 1\beta_0 \setminus (10\alpha_0 \cdot 00\alpha_0) = 10\alpha_0 \cdot 00\alpha_0; \\
(\gamma \backslash \alpha) \setminus (\gamma \backslash \beta) = (10\alpha_0 \cdot 00\alpha_0) \setminus 1\beta_0 = e\beta_0, \\
(\alpha \backslash \gamma) \setminus (\alpha \backslash \beta) = \phi_0 \cdot 1\beta_0 = e\beta_0,
\]
and we are done.

Case 3.4.2. The address \( \beta \) is equal to 1. Then we find 
\[
(\alpha \backslash \beta) \setminus (\alpha \backslash \gamma) = 1 \backslash \phi = \phi \cdot 1 \cdot 0, \\
(\beta \backslash \alpha) \setminus (\beta \backslash \gamma) = 0\alpha_0 \setminus (\phi \cdot 1 \cdot 0) = \phi \cdot 1 \cdot 0, \\
(\beta \backslash \gamma) \setminus (\beta \backslash \alpha) = (\phi \cdot 1 \cdot 0) \setminus 0\alpha_0 = 110\alpha_0 \cdot 100\alpha_0 \cdot 010\alpha_0 \cdot 000\alpha_0, \\
(\gamma \backslash \beta) \setminus (\gamma \backslash \alpha) = (1 \cdot \phi) \setminus (10\alpha_0 \cdot 00\alpha_0) = 110\alpha_0 \cdot 010\alpha_0 \cdot 100\alpha_0 \cdot 000\alpha_0, \\
\]
and we find 110\alpha_0 \cdot 010\alpha_0 \cdot 000\alpha_0 \equiv 1^+ 110\alpha_0 \cdot 010\alpha_0 \cdot 100\alpha_0 \cdot 000\alpha_0.

Finally, we have 
\[
(\gamma \backslash \beta) \setminus (\beta \backslash \gamma) = (10\alpha_0 \cdot 00\alpha_0) \setminus (1 \cdot \phi) = 1 \cdot \phi, \\
(\alpha \backslash \gamma) \setminus (\alpha \backslash \beta) = \phi \cdot 1 = 1 \cdot \phi,
\]
and we are done for this case.

Case 4. The greatest common prefix of two addresses is a prefix of the third address. We assume that the greatest common prefix of \( \alpha \) and \( \beta \) is a prefix of \( \gamma \). If \( \alpha \perp \beta \) holds, there exists \( \delta \) satisfying \( \delta 0 \subset \alpha \) and \( \delta 1 \subset \beta \), or conversely. Then we have either \( \delta 0 \subset \gamma \), and, then, \( \beta \) is orthogonal to the greatest common prefix of \( \alpha \) and \( \gamma \), or \( \delta 1 \subset \gamma \) and \( \alpha \) is orthogonal to the greatest common prefix of \( \beta \) and \( \gamma \). In both cases, we are in case 2 above. Finally, if \( \alpha \perp \beta \) fails, we may assume \( \beta \subset \alpha \). Then \( \beta \subset \gamma \) necessarily holds, and, therefore, \( \beta \) is a prefix of the greatest common prefix of \( \alpha \) and \( \gamma \), and we are in case 3 above. So the proof is complete.

By Proposition II.2.1 (coherent), we deduce:

Proposition 1.10. (completeness) Word reversing is complete for \( f \), i.e., for all words \( u, u' \) on \( A \), \( u \equiv^+ u' \) is equivalent to \( u^{-1}u' \cap \varepsilon \).

The next step is to prove that word reversing using \( f \) always converges.

Proposition 1.11. (convergence) The complement \( f \) is convergent. If \( u, v \) are words on \( A \) and their cumulated lengths in 0’s and 1’s is \( \ell \), reversing of \( u^{-1}v \) requires at most \( \exp^\ast(O(\ell)) \) steps.
Section VIII.1: The Group $G_{LD}$ and the Monoid $M_{LD}$

Proof. (We recall that $\exp^*(n)$ is a base 2 tower of exponentials of height $n$.)

For convergence, by Proposition II.2.16 (lcm), it suffices that we show that, for all positive words $u$, $v$, there exist positive words $u'$, $v'$ satisfying $uv' \equiv^+ v' u$. This follows from Proposition VII.3.22 (syntactic confluence).

For complexity, Proposition VII.3.22 tells us that $\partial^\ell t_{u,v}$ represents a common right multiple of the classes of $u$ and $v$ in $M_{LD}$. An induction shows that, when the cumulated length of $u$ and $v$ in terms of 0's and 1's is $\ell$, then the size of the term $t_{u,v}'$ lies in $O(\ell)$. Hence the size of $\partial^\ell t_{u,v}$ is bounded above by $\exp^*(O(\ell))$. We conclude using Exercise II.3.23 (complexity).

It follows that the results of Chapter II apply to the monoid $M_{LD}$. We obtain:

Proposition 1.12. (left cancellative) The monoid $M_{LD}$ is left cancellative.

Proposition 1.13. (lattice) Every finite subset of $M_{LD}$ admits a right lcm and a left gcd, and $M_{LD}$ equipped with the operations $\lor$ and $\land$ is a lattice.

Proposition 1.14. (fraction) (i) We have $G_{LD} = (G_{LD}^+)(G_{LD}^+)^{-1}$, i.e., every element of $G_{LD}$ admits an expression of the form $uv^{-1}$ with $u$, $v$ words on $A$.

(ii) For $w$, $w'$ words on $A \cup A^{-1}$, the relation $w \equiv w'$ holds, i.e., $w$ and $w'$ represent the same element of $G_{LD}$, if and only if there exist positive words $v$, $v'$ satisfying

$$N(w) v \equiv^+ N(w') v' \quad \text{and} \quad D(w) v \equiv^+ D(w') v'.$$

(iii) In particular, for $w$, $w'$ words on $A$, $u \equiv u'$ holds if and only if $uw \equiv^+ u'w$ holds for some word $v$ on $A$. 

Figure 2.1. Coherence of the complement $f$
A natural question is whether \( M_{LD} \) is also associated with a complement on the left. The answer is positive, but the involved complement fails to be left coherent. Actually, left lcm’s do not always exist in \( M_{LD} \), and it is false that every element of \( G_{LD} \) can be expressed as a left fraction (see Exercise 1.20).

\[\text{Exercise 1.15. (kernel)}\] Prove that the kernel of the projection of \( G_{LD} \) onto \( B_{\infty} \) is the normal subgroup of \( G_{LD} \) generated by the elements of the form \( g_{0\alpha} \). [Hint: Every generator \( g_{1\alpha} \) is conjugated in \( G_{LD} \) with some generator \( g_{0\alpha} \), as we have \( g_{1\alpha} = g_{-1\alpha} \cdots g_{0\alpha} g_{-1\alpha}^{-1} \cdots g_{1\alpha}^{-1} \).]

\[\text{Exercise 1.16. (other presentation)}\] (i) Write \( a_i = g_{1i-1} \). Prove that \( G_{LD} \) is generated by the sequence \( a_1, a_2, \ldots \), and that \( a_i a_j = a_j a_i \) holds for \( |i - j| \geq 2 \). [Hint: For \( \alpha = 01^j \), prove, using relations of type 10, \( g_{\alpha} = g_{0\beta}^{-1} g_{31\beta}^{-1} \cdots g_{31\beta}^{-1} g_{0\beta} g_{31\beta}^{-1} \cdots g_{0\beta} g_{31\beta}^{-1} \).]

\[\text{Exercise 1.17. (abelianization)}\] Prove that the abelianization \( G_{LD}^{ab} \) of \( G_{LD} \) is obtained by adding the relations \( g_{0\alpha} = g_{0\alpha}, g_{31\alpha} = g_{31\alpha} \). Deduce that \( G_{LD}^{ab} \) is isomorphic to \( \mathbb{Z}[z] \), the image of \( g_{10} \) in \( G_{LD}^{ab} \) being \( z^1 \).

\[\text{Exercise 1.18. (braidlike)}\] Say that a word on \( A \) is braided if it only contains addresses of the form \( 1^m \). Prove that, for every word \( u \) on \( A \), there exist words \( v, u_0, u_1, \ldots \) on \( A \) such that \( v \) is braided and \( u^1 v = 1^0 u_1 \).

\[\text{Exercise 1.19. (no minimal section)}\] Prove that there exists no minimal section for the projection of \( M_{LD} \) onto \( B_{\infty}^{+} \) in the sense that there is no section \( s \) of \( pr(a) \) such that every element \( a \) of \( M_{LD} \) is a right multiple of \( s(pr(a)) \). [Hint: Consider \( a = g_{4\sigma_1} g_{4\sigma_2} g_{4\sigma_3} \), which projects onto \( \sigma_1 \sigma_2 \sigma_1 \). Show that \( s(\sigma_1 \sigma_2 \sigma_1) \) should be one of \( a, g_{4\sigma_1} g_{4\sigma_2} g_{4\sigma_3} \), or \( g_{4\sigma_2} g_{4\sigma_1} g_{4\sigma_3} \), and that none of these elements is equivalent.]
Exercise 1.20. (left complement) (i) Prove there exists a unique complement \( f_L \) on \( A \) such that \( M_{LD} \) is associated with \( f_L \) on the left.

(ii) Prove that the domain of the operator \( LD_\phi \cdot LD_0 \cdot LD_\phi^{-1} \) is empty. Deduce that no equality \( u_1 \cdot \phi \cdot 0 \equiv^* v_1 \cdot \phi \) with \( u_1, v_1 \) positive words may hold. Conclude that the left reversing of the word \( 1 \cdot 10 \cdot 1^{-1} \) (using the complement \( f_L \)) does not terminate.

(iii) Consider \( u = 1 \cdot \phi \cdot 01, \ u' = 1 \cdot 10 \cdot \phi, \ v = 1. \) Check that the left reversing of \( uv^{-1} \) succeeds, leading to the word \( 1 \cdot 10 \cdot 1^{-1} \cdot \phi^{-1} \cdot 1 \cdot \phi \cdot 0 \). Conclude that the complement \( f_L \) is not left coherent.

(iv) Check the equivalence \( \phi \cdot 1 \cdot 10 \cdot \phi \equiv^* 1 \cdot \phi \cdot 01 \cdot 1. \) Show that the left reversing of \( (\phi \cdot 1 \cdot 10 \cdot \phi)(1 \cdot \phi \cdot 01 \cdot 1)^{-1} \) does not lead to the empty word. Conclude that left word reversing is not complete for \( f_L \).

(v) Use the previous results to prove the existence of a pair \( (a, b) \) in \( M_{LD} \) such that \( a \) and \( b \) have no left lcm, and the existence of an element in \( G_{LD} \) that cannot be expressed as a left fraction \( a^{-1}b \) with \( a, b \) in \( G_{LD}^+ \).

Exercise 1.21. (right cancellation) Assume that \( u, u' \) are words on \( A \) that satisfy \( u \cdot 1^i \equiv^* u' \cdot 1^i. \) Prove that the braid words \( \text{pr}(u) \) and \( \text{pr}(u') \) are equivalent, and conclude that we must have \( \text{pr}(u/u') = \text{pr}(u'/u) = \varepsilon. \) [Hint: Use Lemma 1.6.] Deduce that there exist positive words \( u_0, v_1, \ldots \) and \( v \) satisfying \( uu' = \prod_i 1^0u_i, \) and \( 1^i \equiv^* \prod_i 1^0u_i \cdot v. \) Conclude that \( u \equiv^* u' \) holds.

2. The Blueprint of a Term

Among the specific properties of left self-distributivity, the phenomenon called absorption by right powers in Chapter V plays a significant role. As every result stating that two terms \( t, t' \) are LD-equivalent, it admits a syntactic counterpart consisting in explicitly computing a sequence of addresses, i.e., a word on \( A \cup A^{-1}, \) say \( w(t, t') \), such that \( t' \) is obtained from \( t \) by applying left self-distributivity successively at the addresses indicated in \( w(t, t') \). In the current case, we consider the equivalence

\[
x^{[p+1]} =_{LD} t \cdot x^{[p]}
\]

which we know holds for \( p \) large enough for every term \( t \) in \( T_1. \) Here we show that some corresponding word \( w(x^{[p+1]}, t \cdot x^{[p]}) \) can be chosen so as to depend
on $t$ only, and that this word can be used as a syntactic copy of $t$ on which we can simulate many properties. This word—or its class in $G_{\mathrm{LD}}$—will be called the blueprint of $t$. Studying such blueprints will lead us to the braid exponentiation of Chapter I, as well as to a complete description of the connection between the monoid $G_{\mathrm{LD}}$ and the group $G_{\mathrm{LD}}$.

**The syntactic content of absorption by right powers**

In order to find a definition for the blueprint of a term informally introduced above, it suffices to consider the inductive proof of Formula (2.1) given in Section V.2. For $t = x$, (2.1) is an equality. Otherwise, for $t = t_1 t_2$, and $p$ large enough, we have written

$$x^{[p+1]} =_{\mathrm{LD}} t_1 \cdot x^p =_{\mathrm{LD}} t_1 \cdot (t_2 \cdot x^{[p-1]}) =_{\mathrm{LD}} (t_1 t_2) \cdot (t_1 \cdot x^{[p-1]}) =_{\mathrm{LD}} t \cdot x^p.$$  

(2.2)

Assume that $f_1$ and $f_2$ are operators in $G_{\mathrm{LD}}$ such that, for $p$ large enough, $f_1$ maps $x^{[p+1]}$ to $t_1 \cdot x^p$, and similarly, $f_2$ maps $x^{[p+1]}$ to $t_2 \cdot x^p$. Then (with an obvious meaning) $\chi_1(f_2)$ maps $t_1 \cdot x^p$ to $t_1 \cdot (t_2 \cdot x^{[p-1]})$, and $\chi_1(f_1^{-1})$ maps $t \cdot (t_1 \cdot x^{[p-1]})$ to $t \cdot x^p$. Now, by definition, $\mathrm{LD}_{\phi}$ maps $t_1 \cdot (t_2 \cdot x^{[p-1]})$ to $(t_1 t_2) \cdot (t_1 \cdot x^{[p-1]})$, i.e., $t \cdot (t_1 \cdot x^{[p-1]})$. By composing the previous operators, we conclude that

$$f_1 \cdot \chi_1(f_2) \cdot \mathrm{LD}_{\phi} \cdot \chi_1(f_1^{-1})$$

(2.3)

maps $x^{[p+1]}$ to $t \cdot x^p$. Formula (2.3), which is nothing more than a syntactic translation of (2.2), tells us how to define the blueprint of a term inductively:

**Definition 2.1.** (blueprint) For $t$ a term in $T_1$, the blueprint of $t$ is the word $\chi_t$ inductively defined by $\chi_x = \varepsilon$ and

$$\chi_{t_1 t_2} = \chi_{t_1} \cdot 1 \chi_{t_2} \cdot \phi \cdot 1 \chi_{t_1}^{-1}.$$  

(2.4)

**Example 2.2.** (blueprint) Let $t = (x \cdot x) \cdot x$. We have $\chi_x = \varepsilon$, hence $\chi_{(x \cdot x)} = \varepsilon \cdot \varepsilon \cdot \phi \cdot \varepsilon^{-1} = \phi$, and, finally, we find $\chi_t = \phi \cdot \varepsilon \cdot \phi \cdot \varepsilon^{-1} = \phi \cdot \phi \cdot 1^{-1}$.

By construction, we have:

**Proposition 2.3.** (effective absorption) For every term $t$ in $T_1$, the operator $\mathrm{LD}_{\chi_t}$ maps $x^{[p+1]}$ to $t \cdot x^p$ for $p \geq \mathrm{ht}(t)$.

Here we arrive at the key point. Assume that $t$ and $t'$ are LD-equivalent terms. Then there exists a word $w$ on $A \cup A^{-1}$ such that $\mathrm{LD}_w$ maps $t$ to $t'$. Now, by Proposition 2.3 (effective absorption), the operator $\mathrm{LD}_{\chi_t}$ maps $x^{[p+1]}$ to $t' \cdot x^p$.
for \( n \) large enough. Similarly, the operator \( \text{LD}_x \) maps \( x^{[p+1]} \) to \( t.x^{[p]} \). As \( \text{LD}_w \) maps \( t \) to \( t' \), the operator \( \text{LD}_{0w} \) maps \( t.x^{[p]} \) to \( t'.x^{[p]} \), and we have found two operators that map \( x^{[p+1]} \) to \( t'.x^{[p]} \), namely \( \text{LD}_{x'} \) and \( \text{LD}_{x'} \cdot \text{LD}_{0w} \). We have seen in Chapter VII that, if \( u \) and \( u' \) are positive words and the operators \( \text{LD}_u \) and \( \text{LD}_{u'} \) take the same value on at least one term, \( i.e. \), if \( t \cdot u = t' \cdot u' \) holds for at least one term \( t \), then \( \text{LD}_u \) and \( \text{LD}_{u'} \) coincide. We did not prove that this implies \( u \equiv u' \), neither did we prove a similar result for arbitrary words, and we have so far no way for deducing the relation \( w \equiv w' \) from a possible equality \( \text{LD}_w = \text{LD}_{w'} \). However, the definition of \( \equiv \) has been made by considering those relations that we know are satisfied by the operators \( \text{LD}_w \). So, if our analysis is correct, we can expect that an equality of the form \( t \cdot w = t' \cdot w' \) implies \( w \equiv w' \), and, therefore, we can expect that

\[
\chi_{t \cdot w} \equiv \chi_{t} \cdot \chi_{1}.
\]  

(2.5)

holds. The point is that, if this equivalence is true, then, necessarily, it can be established by a direct verification, and this is what we are going to do now.

To this end, it is convenient that we introduce the binary operation \( \wedge \) on \((A \cup A^{-1})^*\) such that \( \chi_{t_1 \cdot t_2} = \chi_{t_1} \wedge \chi_{t_2} \).

**Definition 2.4.** (exponentiation) For \( u, v \) words on \( A \cup A^{-1} \), we define

\[
u \wedge v = u \cdot 1 \cdot v \cdot \phi \cdot 1.u^{-1}.
\]  

(2.6)

With this notation, the definition of the blueprint can be restated as:

**Lemma 2.5.** The mapping \( \chi \) is the homomorphism of \( T_1 \) into \((A \cup A^{-1})^*, \wedge \) that maps \( x \) to \( e \), \( i.e. \), \( \chi_t \) is \( t(e) \) evaluated in \((A \cup A^{-1})^*, \wedge \).

Let us consider (2.5) in the case \( w = \phi \). Assuming \( t' = t \cdot \phi \) means that there exist three terms \( t_1, t_2, t_3 \) satisfying \( t = (t_1 \cdot t_2 \cdot t_3) \) and \( t' = (t_1 \cdot t_2) \cdot (t_1 \cdot t_3) \). So, proving (2.5) here amounts to proving the equivalence

\[
(u \wedge v) \wedge (u \wedge w) \equiv u \wedge (v \wedge w) \cdot 0
\]  

(2.7)

with \( u = \chi_{t_1}, v = \chi_{t_2}, \) and \( w = \chi_{t_3} \). Now, this is the matter of a simple verification:

**Lemma 2.6.** Let \( u, v, w \) be arbitrary words on \( A \cup A^{-1} \). Then (2.7) holds.

**Proof.** Applying the definition and LD-relations, we find

\[
(u \wedge v) \wedge (u \wedge w) \equiv u \cdot 1 \cdot v \cdot \phi \cdot 1.u^{-1} \cdot 1.u \cdot 1.w \cdot 1 \cdot 1.w^{-1} \cdot \phi \cdot 11.u \cdot 1^{-1} \cdot 11.w \cdot 1^{-1} \cdot 11.w^{-1} \cdot 1.u^{-1}
\]

\[
\equiv u \cdot 1 \cdot v \cdot 11.w \cdot \phi \cdot 1 \cdot \phi \cdot 11.w^{-1} \cdot 11.w \cdot 1^{-1} \cdot 11.w^{-1} \cdot 1.u^{-1}
\]

\[
\equiv u \cdot 1 \cdot v \cdot 11.w \cdot \phi \cdot 1 \cdot \phi \cdot 1^{-1} \cdot 11.w^{-1} \cdot 1.u^{-1}
\]
Chapter VIII: The Group of Left Self-Distributivity

\[ \equiv u \cdot 1v \cdot 11w \cdot 1 \cdot \phi \cdot 1 \cdot 0 \cdot 1^{-1} \cdot 11v^{-1} \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot 11w \cdot 1 \cdot \phi \cdot 0 \cdot 11v^{-1} \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot 11w \cdot 1 \cdot 11v^{-1} \cdot \phi \cdot 1u^{-1} \cdot 0 = u \wedge (v \wedge w) \cdot 0. \]

Let us consider now (2.5) when \( w \) consists of a single address of the form \( \alpha = 0\beta \). The hypothesis \( t' = t \alpha \) yields \( t = t_1 \cdot t_2 \), \( t' = t'_1 \cdot t_2 \) with \( t'_1 = t_1 \cdot t \beta \). If we assume, by an induction hypothesis, that \( \chi_{t'_1} = \chi_{t_1 \cdot t \beta} \) holds, then proving (2.5) amounts to proving the equivalence

\[(u \wedge 0w) \wedge v \equiv u \wedge v \cdot 00w \tag{2.8}\]

with \( u = \chi_{t_1}, v = \chi_{t_2} \), and \( w = \beta \). Similarly, proving (2.5) in the case \( \alpha = 1\beta \) amounts to proving a special case of

\[(u \wedge (v \cdot 0w) \equiv u \wedge v \cdot 01w. \tag{2.9}\]

As above, such equivalences can be established by a simple verification.

**Lemma 2.7.** Let \( u, v, w \) be arbitrary words on \( A \cup A^{-1} \). Then (2.8) and (2.9) hold.

**Proof.** We use induction on the length of \( w \). The result is obvious for \( w = \epsilon \). Assume \( \lg(w) = 1 \), say \( w = \alpha^e \) with \( \alpha \) an address and \( e = \pm 1 \). For \( e = +1 \), we obtain

\[(u \cdot 0\alpha) \wedge v = u \cdot 0\alpha \cdot 1v \cdot \phi \cdot 10\alpha^{-1} \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot 0\alpha \cdot \phi \cdot 10\alpha^{-1} \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot \phi \cdot 00\alpha \cdot 10\alpha^{-1} \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot \phi \cdot 00\alpha \cdot 1u^{-1} \equiv u \cdot 1v \cdot \phi \cdot 1u^{-1} \cdot 00\alpha \equiv (u \wedge v) \cdot 00\alpha \]

as was expected. For \( e = -1 \), applying the previous result to \( u \cdot 0\alpha^{-1} \) gives \((u \cdot 0\alpha^{-1}) \wedge v \equiv (u \cdot 0\alpha^{-1}) \wedge v \cdot 00\alpha \), hence \((u \cdot 0\alpha^{-1}) \wedge v \equiv (u \wedge v) \cdot 00\alpha -1 \). The case \( \lg(w) \geq 2 \) follows from an easy induction.

We prove now (2.9) using a similar induction. Again, the only nontrivial case is when \( w \) consists of a single address, say \( w = \alpha \). Then we obtain

\[(u \wedge (v \cdot 0\alpha) = u \cdot 1v \cdot 10\alpha \cdot \phi \cdot 1u^{-1} \]
\[ \equiv u \cdot 1v \cdot \phi \cdot 01\alpha \cdot 1u^{-1} \equiv u \cdot 1v \cdot 01\alpha \cdot 1u^{-1} \cdot 01\alpha = (u \wedge v) \cdot 01\alpha. \]

We easily deduce a proof of the conjectured equivalence (2.5):

**Proposition 2.8.** (action I) Assume that \( t \) is a term in \( T_1 \), and \( w \) is a word on \( A \cup A^{-1} \) such that \( t \cdot w \) exists. Then we have

\[ \chi_{t \cdot w} \equiv \chi_t \cdot 0w. \tag{2.10} \]
Proof. We use induction on the length of \( w \). For \( w = \varepsilon \), the result is obvious. Assume \( \lg(w) = 1 \), say \( w = \alpha e \) with \( \alpha \) an address and \( e = \pm 1 \). Assume first \( e = +1 \). We use induction on the length of \( \alpha \). For \( \alpha = \emptyset \), the result follows from Lemma 2.6, as was noted above. For \( \alpha = 0 \beta \), it follows from the induction hypothesis and from (2.8) in Lemma 2.7; for \( \alpha = 1 \beta \), it follows similarly from the induction hypothesis and from (2.9) in Lemma 2.7. The argument is the same for \( e = -1 \). Assume now \( \lg(w) \geq 2 \). Write \( w = w_1w_2 \). Applying the induction hypothesis, we find 

\[
\chi t \cdot w \equiv \chi t \cdot w_1 \cdot 0w_2 \equiv \chi t \cdot 0w_1 \cdot 0w_2 = \chi t \cdot 0w.
\]

\( \blacksquare \)

The exponentiation on \( G_{LD} \)

As a first application of the previous results, we show how to construct a left self-distributivity operation on a quotient of \( G_{LD} \). The latter turns out to be closely connected with braid exponentiation.

As the shift mapping \( sh_1 \) on \((A \cup A^{-1})^\ast\) is compatible with the congruence relation \( \equiv \), the exponentiation of (2.6) induces a well defined operation on the group \( G_{LD} \), which we shall denote by \( ^\wedge \) as well. Thus, for \( a, b \) in \( G_{LD} \), we have

\[
a ^\wedge b = a \cdot sh_1(b) \cdot g_0 \cdot sh_1(a^{-1}).
\]

(2.11)

The exponentiation on \( G_{LD} \) is not a left self-distributive operation. However, the main interest of our approach is that we can measure the lack of self-distributivity exactly. Indeed, translating Lemma 2.6 immediately gives:

Lemma 2.9. For all \( a, b, c \in G_{LD} \), we have

\[
(a ^\wedge b) ^\wedge (a ^\wedge c) = (a ^\wedge (b ^\wedge c)) \cdot g_0.
\]

(2.12)

So, in order to construct a possible left self-distributive operation from the exponentiation of \( G_{LD} \), and owing to (2.12), we have to collapse \( g_0 \). The first idea could be to use the normal subgroup of \( G_{LD} \) generated by \( g_0 \), but exponentiation need not be compatible with the corresponding congruence. Now, Lemma 2.7 gives:

Lemma 2.10. For all \( a, b, c, d \) in \( G_{LD} \), we have

\[
(a \cdot sh_0(c)) ^\wedge (b \cdot sh_0(d)) = (a ^\wedge b) \cdot sh_{00}(c) \cdot sh_{01}(d).
\]

(2.13)

We are thus led to consider right cosets associated with the subgroup \( sh_0(G_{LD}) \) of \( G_{LD} \).

Proposition 2.11. (self-distributive operation) Let \( G_0 \) be the subgroup of \( G_{LD} \) generated by all elements \( g_{0\alpha} \). Then the exponentiation of \( G_{LD} \) induces a left self-distributive operation on the homogeneous set \( G_{LD}/G_0 \).
Chapter VIII: The Group of Left Self-Distributivity

Proof. Write $a \sim a'$ for $a^{-1}a' \in G_0$. Formula (2.13) shows that exponentiation on $G_{LD}$ is compatible with $\sim$, and (2.12) gives $(a^\circ b)^\circ (a^\circ c) \equiv a^\circ (b^\circ c)$ for all elements $a, b, c$ of $G_{LD}$.

We obtain in this way a natural explanation for the existence of the left self-distributive exponentiation of braids. Indeed, we have $B_\infty = G_{LD}/\hat{G}_0$, where $\hat{G}_0$ is the normal subgroup of $G_{LD}$ generated by $G_0$ (Exercise 1.15). So braid exponentiation is the operation induced on $G_{LD}/\hat{G}_0$ by the above left self-distributive operation on $G_{LD}/G_0$, which itself is nothing but the translation of the absorption property of monogenic LD-systems.

The connection between $G_{LD}$ and $G_{rLD}$

The next application is a complete description of the connection between the monoid $G_{LD}$ and the group $G_{rLD}$.

To this end, Formula (2.10) can be used directly (see Exercise 2.28). However, a more simple proof can be obtained by using an alternative formula where the shift mapping $sh_0$ does not appear. This new formula is reminiscent of the braid equality (III.1.3)

$$\prod^x(\vec{a} \cdot b) = (\prod^x \vec{a}) \cdot b \quad (2.14)$$

which connects the action of braids on sequences of braids with a right multiplication—(2.10) also establishes such a connection, but with the endomorphism $sh_0$ added.

Definition 2.12. (half-blueprint) For $t$ in $T_1$, the half-blueprint of $t$ is the word $\chi^*_t$ inductively defined by $\chi^*_x = \varepsilon$ and

$$\chi^*_{t_1 \cdot t_2} = \chi_{t_1} \cdot 1\chi^*_{t_2}. \quad (2.15)$$

Exponentiation in $(A \cup A^{-1})^*$ is reminiscent of a twisted conjugacy, and replacing $\chi_t$ with $\chi^*_t$ amounts to replacing conjugacy with half-conjugacy as defined in Chapter I. Observe for instance that $\chi^*_x[n]$ is the empty word for every $n$. In every case, the word $\chi^*_t$ is a prefix of the word $\chi_t$, essentially its first half; the precise connection between $\chi_t$ and $\chi^*_t$ is as follows:

Lemma 2.13. For $t$ a term in $T_1$ of right height $r$, we have

$$\chi_t \equiv \chi^*_t \cdot \hat{\phi}(r) \cdot 1\chi^{*-1}_t. \quad (2.16)$$

Proof. We use induction on $t$. For $t = x$, the result is obvious. Assume $t = t_1 \cdot t_2$. Using the induction hypothesis and a LD-relation of type 11, we find

$$\chi_t = \chi_{t_1} \cdot 1\chi_{t_2} \cdot \hat{\phi} \cdot 1\chi^{-1}_{t_1}$$
The counterpart to Formula (2.10) involving half-blueprints is as follows:

**Proposition 2.14.** (action II) Assume that $t$ is a term in $T_1$ and $w$ is a word on $A \cup A^{-1}$ such that $t \cdot w$ exists. Then we have

$$
\chi^*_t \cdot w = \chi^*_t \cdot w.
$$

*(Proof)* We use induction on the length of $w$. As above, the point is the case when $w$ consists of a single positive address, say $w = \alpha$. Let us write $t = t_1 \cdot t_2 \cdots$, and let $w = \chi_{t_1}$. There are two cases, according to whether $\alpha$ has the form $1^i$ or not. Assume first $\alpha = 1^{i-1}0 \beta$. Applying the definition, Proposition 2.8 (action I), and LD-relations of type $\perp$, we find

$$
\chi^*_t \cdot w = w_1 \cdot 1^{i-1} \chi_{t_1} \cdot 1^{i} w_{i+1} \cdots \\
\equiv w_1 \cdot 1^{i-1} w_i \cdot 1^{i-1}0 \beta \cdot 1^{i} w_{i+1} \cdots \\
\equiv w_1 \cdot 1^{i-1} w_i \cdot 1^{i-1} w_{i+1} \cdots \cdot 1^{i-1}0 \beta = \chi^*_t \cdot 1^{i-1}0 \beta.
$$

Assume now $\alpha = 1^{i-1}$. We find

$$
\chi^*_t \cdot w = w_1 \cdot 1^{i-1} \chi_{t_1} \cdot 1^{i} w_{i+1} \cdots \\
\equiv w_1 \cdot 1^{i-1} w_i \cdot 1^{i-1} w_{i+1} \cdot 1^{i-1} \cdot 1^{i-1} w_{i+2} \cdots \\
\equiv w_1 \cdot 1^{i-1} w_i \cdot 1^{i-1} w_{i+1} \cdot 1^{i-1} \cdot 1^{i-1} w_{i+2} \cdots \\
\equiv w_1 \cdot 1^{i-1} w_i \cdot 1^{i-1} w_{i+1} \cdot 1^{i-1} w_{i+2} \cdots \cdot 1^{i-1}.
$$

Describing the connection between $G_{LD}$ and $G_{\perp LD}$ is now easy.

**Proposition 2.15.** (connection) Assume that $w$, $w'$ are words on $A \cup A^{-1}$, and the domains of $LD_w$ and $LD_{w'}$ are not disjoint. Then the following are equivalent:

(i) There exists at least one term $t$ in $T_1$ satisfying $t \cdot w = t \cdot w'$;

(ii) For every $t$, we have $t \cdot w = t \cdot w'$ when the latter terms exist;

(iii) The relation $w \equiv w'$ holds.

In particular, for $u$, $u'$ positive words, $LD_u = LD_{u'}$ is equivalent to $u \equiv u'$.

*(Proof)* Assume that both $LD_w$ and $LD_{w'}$ both map $t$ to $t'$. By Proposition 2.14 (action II), we have

$$
\chi^*_t \cdot w = \chi^*_t \cdot w' = \chi^*_{t'} \cdot w',
$$

hence $w \equiv \chi^*_t \equiv \chi^*_{t'} \equiv w'$. 

---

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Conversely, assume that \( w \equiv w' \) holds, and both \( t \cdot w \) and \( t \cdot w' \) exist. By Lemma 2.8, the term \( t \) also belongs to the domain of the operators \( \text{LD}_{N(w)} \cdot \text{LD}_{D(w\backslash \{\emptyset\})}^{-1} \) and \( \text{LD}_{N(w')} \cdot \text{LD}_{D(w')}^{-1} \). By Proposition 1.14 (fraction), there exists two positive words \( u, u' \) satisfying

\[
N(w) \cdot u \equiv^+ N(w') \cdot u' \quad \text{and} \quad D(w) \cdot u \equiv^+ D(w') \cdot u'.
\] (2.18)

Then we have

\[
t \cdot w = t \cdot (N(w) \cdot D(w)^{-1}) = t \cdot (N(w) \cdot v \cdot v^{-1} \cdot D(w)^{-1})
= t \cdot (N(w') \cdot v' \cdot v'^{-1} \cdot D(w')^{-1}) = t \cdot (N(w') \cdot D(w')^{-1}) = t \cdot w'.
\]

In the case of positive words \( u, u' \), the intersection of the domains of \( \text{LD}_{u} \) and \( \text{LD}_{u'} \) is never empty, and, by Proposition VII.1.26 (compatibility), the conditions (i) and (ii) are equivalent to \( \text{LD}_{u} = \text{LD}_{u'} \).

It would be tempting to introduce the relation \( \sim \) on \( \mathcal{G}_{\text{LD}} \setminus \{\emptyset\} \) such that \( f \sim f' \) holds if there exists at least one term \( t \) satisfying \( f(t) = f'(t) \), and to try to prove that \( \mathcal{G}_{\text{LD}} \setminus \{\emptyset\} \) quotiented by \( \sim \) embeds in \( \mathcal{G}_{\text{LD}} \). Unfortunately, this does not make sense, as \( \sim \) is not a transitive relation: consider for instance \( f = \text{LD}_{0^{-1}.e^{-1}.\emptyset.0}, f' = \text{LD}_{e}, \) and \( f'' = \text{LD}_{\emptyset.0.\emptyset} \); then \( f, f', \) and \( f'' \) are the identity mappings of their respective domains, the domains of \( f \) and \( f' \) intersect, and so do the domains of \( f' \) and \( f'' \), but the domains of \( f \) and \( f'' \) are disjoint, as \( \text{LD}_{\emptyset.0.\emptyset} \) is empty. Such problems however cannot occur when we restrict to positive transformations, as the intersections of domains are never empty. Thus, we deduce from Proposition 2.15:

**Proposition 2.16. (embedding)** The monoid \( \mathcal{G}_{\text{LD}}^+ \) is isomorphic to the sub-monoid \( G_{\text{LD}}^{+} \) of \( \mathcal{G}_{\text{LD}} \).

We conclude this section with an application. As the complement \( f \) is coherent and convergent, the word problem of the monoid \( M_{\text{LD}} \) is decidable: two positive words \( u, u' \) represent the same element of \( M_{\text{LD}} \) if and only if reversing \( u^{-1} \cdot u' \) ends with an empty word. As we have not proved that the monoid \( M_{\text{LD}} \) is right cancellative, we cannot resort to the results of Chapter II to solve the word problem for the group \( G_{\text{LD}} \). However, the previous result gives a solution.

**Proposition 2.17. (word problem)** The group \( G_{\text{LD}} \) has a solvable word problem.

**Proof.** Assume that \( w \) is a word on \( A \cup A^{-1} \). Then \( w \equiv e \) is equivalent to \( N(w) \equiv D(w) \). By Proposition 2.15 (connection), the latter condition is equivalent to \( \text{LD}_{N(w)} = \text{LD}_{D(w)} \). This equality can be decided effectively: indeed, we have seen in Chapter VII how to determine the terms \( t_{u}^{n} \) and \( t_{w}^{n} \) for every word \( u, \) and \( \text{LD}_{N(w)} = \text{LD}_{D(w)} \) is equivalent to the conjunction of \( t_{N(w)}^{u} = t_{D(w)}^{u} \) and \( t_{N(w)}^{w} = t_{D(w)}^{w} \).
The action of $G_{LD}$ on terms

Associating with every word $w$ on $A \cup A^{-1}$ the operator $LD_w$ defines a partial action of $(A \cup A^{-1})^*$ on terms. It follows from Proposition 2.15 (equivalence) that this action induces a well defined partial action of the group $G_{LD}$. We extend our notation in the natural way: for $t$ a term and $a$ an element $G_{LD}$, we define $t \cdot a$ to be the term $t \cdot w$ where $w$ is an expression of $a$ such that $t \cdot w$ exists, if such an expression exists. By Lemma 2.8, if $t \cdot w$ exists, so does $t \cdot N(w)D(w)^{-1}$, and, if the latter exists, so does $t \cdot N(w')D(w')^{-1}$ for every word $w'$ satisfying $w' \equiv w$. Thus $t \cdot a$ exists if and only if $t \cdot w$ does, where $w$ is an arbitrary expression of $a$ as a right fraction.

In this framework, Proposition 2.14 (action II) gives, for every term $t$ in $T_1$ and every $a$ in $G_{LD}$ such that $t \cdot a$ is defined, the equality

\[ \chi^*_t \circ a = \chi^*_t \cdot a \]  \hspace{1cm} (2.19)

which is an exact counterpart of (2.14). In particular, if $x^{[n]} \cdot a$ exists, we have

\[ x^{[n]} \circ a = a. \]  \hspace{1cm} (2.20)

An application of the previous results is that a term of $T_1$ is completely determined by the class of its blueprint in $G_{LD}$.

**Definition 2.18.** (functions $\chi$ and $\chi^*$) For $t$ in $T_1$, the classes of $\chi_t$ and $\chi^*_t$ in $G_{LD}$ are denoted $\chi_t$ and $\chi^*_t$ respectively.

**Proposition 2.19.** (injectivity) The mapping $\chi$ is injective on $T_1$.

**Proof.** Assume $\chi_t \equiv \chi_{t'}$. Then, by Proposition 2.15 (equivalence), the operators $LD_{\chi_t}$ and $LD_{\chi_{t'}}$ agree on every term in the intersection of their domains. By construction, the term $x^{[p+1]}$ belongs to this intersection for $p$ large enough. For such a $p$, we have

\[ x^{[p+1]} \cdot \chi_t = t \cdot x^{[p]}, \quad \text{and} \quad x^{[p+1]} \cdot \chi_{t'} = t' \cdot x^{[p]}, \]

The hypothesis implies $t \cdot x^{[p]} = t' \cdot x^{[p]}$, hence $t = t'$. \[ \square \]

**Definition 2.20.** (special) Assume that $a$ belongs to $G_{LD}$. We say that $a$ is *special* if it belongs to the image of the mapping $\chi$, i.e., if $a$ is the image of a (necessarily unique) term in $T_1$. The set of all special elements in $G_{LD}$ is denoted by $G_{LD}^s$.

The group $G_{LD}$ plays with respect to the braid group $B_{\infty}$ the same role as the absolutely free system $T_1$ plays with respect to the free LD-system $FLD_1$: $B_{\infty}$.
includes a copy $B_{sp}$ of FLD, while $G_{ld}$ includes a copy $G_{sp}^{ld}$ of $T_1$, as shown in the commutative diagram

$$
\begin{array}{ccc}
T_1 & \xrightarrow{\chi} & G_{sp}^{ld} \subseteq G_{ld} \\
\downarrow & & \downarrow \\
F LD_1 & \xrightarrow{=} & B_{sp} \subseteq B_{\infty}
\end{array}
$$

We then have counterparts for the special decompositions of braids. Let us say that the element $a$ of $G_{ld}$ admits a special decomposition if it admits at least one decomposition of the form

$$a = a_1 \cdot \text{sh}_1(a_2) \cdot \text{sh}_{11}(a_3) \cdots$$

(2.21)

where $a_1, a_2, \ldots$ are special. Observe that, if $a_i = \chi_t$, then the product involved in (2.21) is $\chi_t^*$, where $t$ is the term $t_1 \cdot t_2 \cdots$. Thus admitting a special decomposition means lying in the image of $\chi^*$.

**Proposition 2.21.** (special decomposition) Assume that $a$ is an element of $G_{ld}$ such that $LD_a$ is nonempty. Then $a$ admits in $G_{ld}$ the decomposition

$$a = \chi_t^{-1} \chi_{t^* a}$$

(2.22)

where $t$ is an arbitrary term in the domain of $LD_a$. In particular, if $x[n]$ belongs to the domain of $LD_a$, then $a$ admits the special decomposition $a = \chi_{x[n] \cdot t^* a}$.

**Proof.** The results follow from (2.19) and (2.20) directly. ■

The previous result does not apply to those elements $a$ of $G_{ld}$ such that $LD_a$ is empty. However, even such an element admits a decomposition involving special elements, as $a$ can be expressed as $bc^{-1}$ with $b,c \in G_{ld}^+$, hence as $b_1^{-1}b_2c_2^{-1}c_1$ with $b_1, b_2, c_1, c_2$ admitting a special decomposition.

**The case of several variables**

In this section, we show how extending the previous construction to arbitrary terms naturally leads to the group $CB_{\infty}$ of charged braids introduced in Section V.6.

For $t$ in $T_1$, the blueprint $\chi_t$ of $t$ has been defined so that $LD_{\chi_t}$ maps $x[p+1]$ to $t \cdot x[p]$ for $p$ large enough. Thus $\chi_t$ describes a construction of $t$ from $x[p+1]$—we could use the extended term $x^{\infty}$ to have a uniform starting point. As left self-distributivity preserves the set of variables of the terms it is applied to, we cannot extend the definition so as to construct terms with more than one variable. However, we can do it at the expense of introducing new operators that change the variables.
Definition 2.22. (variable shift) Assume that \( \alpha \) is an address. We define the partial variable shift operator \( \text{VS}_\alpha \) on \( T_\infty \) as follows: The term \( t \) belongs to the domain of \( \text{VS}_\alpha \) if and only if \( \alpha \) belongs to the skeleton of \( t \), and, in this case, the image of \( t \) under \( \text{VS}_\alpha \) is obtained from \( t \) by shifting by one unit the indices of all variables occurring below \( \alpha \) in \( t \).

Example 2.23. (operator \( \text{VS}_\alpha \)) Let \( t \) be the term \( x_1 x_2 x_3 x_2 \). Then \( t \) belongs to the domain of the operators \( \text{VS}_{10} \) and \( \text{VS}_{10}^{-1} \). The images are obtained by shifting by +1 and −1 respectively the variables in the 10-subterm of \( t \), so they are respectively

\[
\begin{align*}
\begin{array}{c}
\text{and}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{Observe that every term belongs to the domain of } \text{VS}_{\gamma}, \text{ while the domain of } \text{VS}_{\gamma}^{-1} \text{ consists of those terms where } x_1 \text{ does not occur.}
\end{align*}
\]

Now, the idea is obvious: using the additional operators \( \text{VS}_\alpha \), we can construct from \( x^{[p]} \) not only terms in \( T_1 \), but also arbitrary terms in \( T_\infty \).

The first step is to introduce the completed geometry monoid \( \tilde{G}_{\text{LD}} \) generated by all operators \( \text{LD}_\alpha \) and \( \text{VS}_\beta \) and their inverses. The elements of these monoids are represented by words on \( \mathbb{A} \cup \mathbb{A}^{-1} \), augmented with a new alphabet that represents the \( \text{VS}_\beta \) operators and their inverses. We use \( \tilde{\beta} \) for the letter that corresponds to the operator \( \text{VS}_\beta \). The shift mapping \( \text{sh}_\gamma \) is extended by \( \text{sh}_\gamma (\tilde{\beta}) = \frac{1}{\gamma} \beta \). Then, as in Section VII.2, we look for the geometric relations satisfied by the operators. It is easy to obtain the following list:

\[
\begin{align*}
\text{VS}_\alpha &\approx \text{VS}_{\alpha 1} \cdot \text{VS}_{\alpha 0}, & \text{VS}_\alpha \cdot \text{VS}_\beta = \text{VS}_\beta \cdot \text{VS}_\alpha \\
\text{VS}_{\gamma 0 \alpha} \cdot \text{LD}_{\gamma 1 \beta} = \text{LD}_{\gamma 1 \beta} \cdot \text{VS}_{\gamma 0 \alpha}, & \text{VS}_\gamma \cdot \text{LD}_{\gamma 0 \alpha} = \text{LD}_{\gamma 0 \alpha} \cdot \text{VS}_\gamma, \\
\text{VS}_{\gamma 0 \alpha} \cdot \text{LD}_\gamma = \text{LD}_\gamma \cdot \text{VS}_{\gamma 10 \alpha} \cdot \text{VS}_{\gamma 00 \alpha}, & \text{VS}_{\gamma 10 \alpha} \cdot \text{LD}_\gamma = \text{LD}_\gamma \cdot \text{VS}_{\gamma 01 \alpha}, \\
\text{VS}_{\gamma 11 \alpha} \cdot \text{LD}_\gamma = \text{LD}_\gamma \cdot \text{VS}_{\gamma 11 \alpha}
\end{align*}
\]

in addition to the already known relations between the operators \( \text{LD}_\alpha \)—we use \( f \approx f' \) means that the operator \( f \) and \( f' \) agree on the intersection of their domains. Then, we extend the construction of the blueprint.

Definition 2.24. (blueprint) For \( t \) a term in \( T_\infty \), the blueprint of \( t \) is the word \( \chi_t \) inductively defined by \( \chi_{x_1} = \tilde{0}^{x_1} = 1 \) and \( \chi_{x_1 x_2} = \chi_{x_1} \cdot 1 \chi_{x_2} \cdot \phi \cdot 1 \chi_{x_1}^{-1} \).
For instance, for $t = (x_1 \cdot x_2) \cdot x_3$, we find $\chi_t = \tilde{\gamma}_1 \cdot \tilde{\gamma}_2 \cdot \tilde{\gamma}_1 \cdot \tilde{\gamma}_2 \cdot 1 \cdot \tilde{\gamma}_1^{-1}$.

An immediate induction gives the expected result:

**Lemma 2.25.** Assume that $t$ is a term in $T_\infty$. Then, for $p$ large enough, the operator $\text{LD}_t$ maps $x[p+1]$ to $t \cdot x[p]$.

Like $G_{\text{LD}}$, the completed geometry monoid $\tilde{G}_{\text{LD}}$ is not a group, so we consider instead the group $\tilde{G}_{\text{LD}}'$ for which the relations in (2.23) constitute a presentation. We use $\tilde{g}_\beta$ for the class of $\tilde{\gamma}_\beta$ in $\tilde{G}_{\text{LD}}$. We introduce on $\tilde{G}_{\text{LD}}'$ the exponentiation defined by

$$a ^ \wedge b = a \cdot \text{sh}_1(b) \cdot \tilde{g}_\beta \cdot \text{sh}_1(a^{-1}),$$

and the homomorphism $\chi$ of $T_\infty$ into $(\tilde{G}_{\text{LD}}, ^\wedge)$ that maps $x_i$ to $\tilde{g}_0^{-1}$. Lemmas 2.9 and 2.10 extend to $\tilde{G}_{\text{LD}}'$. Hence, it suffices to collapse the generators $g_0$ to obtain a self-distributive operation on the quotient. Owing to the case of rank 1, we can expect the image of $T_\infty$ in the quotient group to be a free LD-system. Thus, we have to determine what remains from the relations of $\tilde{G}_{\text{LD}}'$ under the collapsing.

Actually, we first observe that all operators $VS_\beta$ are not needed to construct terms. Considering the operators $VS_{1/0}$ is sufficient, as applying the relations of (2.23) lets no other operators appear. We write $\tilde{G}_{\text{LD}}'$ for the subgroup of $\tilde{G}_{\text{LD}}$ generated by all $g_\alpha$ and those generators $\tilde{g}_\beta$ of the above type.

Collapsing the generators $g_0$ of $G_{\text{LD}}$ amounts to projecting $G_{\text{LD}}$ onto the braid group $B_\infty$ by mapping $g_\alpha$ to $\sigma_{i+1}$ for $\alpha = 1'$, and to 1 otherwise. When we consider a similar process in $\tilde{G}_{\text{LD}}'$, the generators $\tilde{g}_{1/0}$ are not collapsed. Let us denote by $\rho_{j+1}$ the image of $\tilde{g}_{1/0}$ in the projection. What remains from (2.23) is the list of all braid relations, augmented with the relations

$$\rho_j \rho_k = \rho_k \rho_j \quad \text{for all } j, k,$$

$$\rho_j \sigma_i = \sigma_i \rho_j \quad \text{for } j < i \text{ or } j > i + 1,$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_i \rho_i \rho_{i+1} \quad \text{for all } i.$$

We recognize the defining relations of $CB_\infty$. So we can state:

**Proposition 2.26.** *(charged braids)* The image of $\tilde{G}_{\text{LD}}'$ under the projection that collapses every generator $g_0$ is the group $CB_\infty$.

It follows that the exponentiation of $\tilde{G}_{\text{LD}}'$ induces a left self-distributive operation on $CB_\infty$—as was already checked in Chapter V—and that the mapping $\chi$ induces a well defined homomorphism of the free LD-system $\text{FLD}_\infty$ into $(CB_\infty, ^\wedge)$ that maps $x_i$ to $\rho_i^{-1}$. The example of rank 1 and ordinary braids suggests that this homomorphism is likely to be an embedding. This result is not obvious, as new relations could have been added when collapsing, but, precisely, it is the content of Proposition V.6.14 (charged braids free).
Thus, the results have been extended from rank 1 to arbitrary rank completely, and, in particular, we have obtained the following commutative diagram

\[
\begin{array}{ccl}
T_\infty & \xrightarrow{X} & G'_{LD} \\
\downarrow & & \downarrow \\
\text{FLD}_\infty & \rightarrow & CB_\infty
\end{array}
\]

where the bottom horizontal arrow is an embedding.

**Exercise 2.27. (abelianization)** Show that using the abelianized group \( G'^{ab}_{LD} = \mathbb{Z}[\mathcal{L}] \) instead of \( G_{LD} \) leads to the blueprint \( \chi^{ab} \) inductively defined by \( \chi^a_2 = 1, \chi^{ab}_{t; t_2} = (1 - z)\chi^{ab}_{t; t_2} + z\chi^{ab}_{t; t_2} + 1. \) Show that collapsing the image of \( g_0 \) amounts to putting \( z = 1 \), and conclude that the left self-distributive operation obtained is \( \mathbb{Z} \), equipped with \( a^\ast b = b + 1 \).

**Exercise 2.28. (left cancellation)** Prove that exponentiation in \( G_{LD} \) is left cancellative.

**Exercise 2.29. (mapping \( \chi^\ast \))** Assume \( t \in T_1 \) and \( \text{ht}_n(t) = r \). Prove that, for \( n \) large enough, \( \text{LD}_\chi^\ast \) maps \( x[b] \) to the term obtained from \( t \) by replacing the rightmost variable with \( x[b-r] \).

**Exercise 2.30. (evaluation)** Let us consider terms in \( T_{(A \cup A^{-1})^r} \) and \( T_{G_{LD}} \), i.e., terms whose variables are themselves words on \( A \cup A^{-1} \) or elements of \( G_{LD} \). For \( t \) in \( T_{(A \cup A^{-1})^r} \), we write \( \text{eval}(t) \) for the evaluation of \( t \) in \( ((A \cup A^{-1})^\ast, \ast) \), i.e., for the element of \( (A \cup A^{-1})^\ast \) iteratively defined by \( \text{eval}(t) = t \) if \( t \) is a variable, and \( \text{eval}(t_1 \cdot t_2) = \text{eval}(t_1) \cdot \text{eval}(t_2) \).

The definition is similar for \( t \in T_{G_{LD}} \).

(i) For \( t \in T_1 \), prove \( \chi_t = \text{eval}(t(\varepsilon)) \), where \( t(\varepsilon) \) denotes the term in \( T_{(A \cup A^{-1})^1} \) obtained by substituting \( x \) with \( \varepsilon \).

(ii) For \( t \) in \( T_{(A \cup A^{-1})^1} \), and \( w \) in \( A \cup A^{-1} \) such that \( t \cdot w \) exists, prove \( \text{eval}(t \cdot w) \equiv \text{eval}(t) \cdot 0w \). [Hint: Use induction on the length of \( w \), as in the proof of Proposition 2.8 (action I).] Re-deduce Proposition 2.8.

(iii) For \( t \) in \( T_{(A \cup A^{-1})^1} \), say \( t = t_1 \cdot \cdots \cdot t_p \cdot x \) with \( x \) a variable, define \( \text{eval}^\ast(t) = \text{eval}(t_1) \cdot 1\text{eval}(t_2) \cdot 1\text{eval}(t_3) \cdot \cdots \cdot 1\text{eval}(x) \). Prove \( \chi_t^\ast = \text{eval}^\ast(t(\varepsilon)) \) for \( t \in T_1 \).

(iv) For \( t \) in \( T_{(A \cup A^{-1})^1} \), and \( w \) in \( (A \cup A^{-1})^\ast \) such that \( t \cdot w \) exists, prove \( \text{eval}^\ast(t \cdot w) \equiv \text{eval}^\ast(t) \cdot w \). Re-deduce Proposition 2.14 (action II).

(v) For \( t \) in \( T_{G_{LD}} \) and \( a \) in \( G_{LD} \) such that \( t \cdot a \) exists, prove \( \text{eval}^\ast(t \cdot a) = \text{eval}^\ast(t) \cdot a \), where \( \text{eval}^\ast(t) \) is defined as in (iii).
Exercise 2.31. (embedding) Prove that $\chi$ is injective on $T_\infty$.

3. Order Properties in $G_{LD}$

We shall see now how the braid order of Chapter III originates in some natural preorder on $G_{LD}$. Actually, we introduce below two different (pre)orders on $G_{LD}$, both relying on its action on terms by left self-distributivity. One consists in comparing $a$ and $b$ by analysing the action of the (non-uniquely defined) numerators and denominators of $ab^{-1}$ on iterated left subterms, and it results in a right invariant preorder connected with the braid order, which gives a direct proof that left division in free LD-systems has no cycle, and a new solution for the word problem of Identity (LD).

The second order on $G_{LD}$ resorts to a linear ordering of terms, and consists in comparing $a$ and $b$ by using their action on terms, as was done for braids in Chapter III; it results in a two-sided invariant linear order on $G_{LD}$. This order can be interpreted in terms of an action of $G_{LD}$ on the Cantor line.

A preordering on $G_{LD}$

We have established in Section 3 that $G_{LD}$ includes a copy $G_{sp}^{LD}$ of $T_1$. The latter set is equipped with a preordering $\subseteq_{LD}$. A natural idea is to define on $G_{LD}$ a relation that includes this copy of $\subseteq_{LD}$.

Let us start from the proof of the confluence property as given in Chapter V. Assume that $t$, $t'$ are terms in $T_1$. Let $w = \chi t^{-1} \cdot \chi t'$. By construction, $LDw$ maps $t \cdot x[p]$ to $t' \cdot x[p]$ for $p$ large enough. Then, we have $(t \cdot x[p]) \cdot N(w) = (t' \cdot x[p]) \cdot D(w)$, and the latter term, say $t_0$, is a common LD-expansion of $t \cdot x[p]$ and $t' \cdot x[p]$. By Lemma VII.4.5, we have

$$t \cdot N(w) = \text{left}(t \cdot x[p]) \cdot N(w) = \text{left}^{dil(1,N(w))}(t_0),$$

$$t' \cdot D(w) = \text{left}(t' \cdot x[p]) \cdot D(w) = \text{left}^{dil(1,D(w))}(t_0).$$

In particular, the term $t$ is LD-equivalent to $\text{left}^{dil(1,N(w))}(t_0)$ and $t'$ is LD-equivalent to $\text{left}^{dil(1,D(w))}(t_0)$. Hence $t \subseteq_{LD} t'$, $t =_{LD} t'$, or $t \supseteq_{LD} t'$ holds according to whether $\text{dil}(1,D(w)) < \text{dil}(1,N(w))$, $\text{dil}(1,D(w)) = \text{dil}(1,N(w))$, or $\text{dil}(1,D(w)) > \text{dil}(1,N(w))$. 

or \(\text{dil}(1, D(w)) > \text{dil}(1, N(w))\) does. This leads us to introduce the following partition on \((A \cup A^{-1})^*\):

**Definition 3.1.** (sets \(P_+, P_0, P_-\)) Assume that \(w\) is a word on \(A \cup A^{-1}\). We say that \(w\) belongs to \(P_+\) (resp. \(P_0\), resp. \(P_-\)) if we have

\[
\text{dil}(1, D(w)) < \text{dil}(1, N(w)) \quad (\text{resp.} \quad =, \quad \text{resp.} \quad >). \tag{3.1}
\]

**Lemma 3.2.** The sets \(P_+, P_0, P_-\) form a partition of \((A \cup A^{-1})^*\); each of them is saturated under \(\equiv\) and closed under product. Moreover, we have \(P_0 P_+ \subseteq P_+, P_+ P_0 \subseteq P_+, P_0 P_- \subseteq P_-,\) and \(P_- P_0 \subseteq P_-\).

**Proof.** Assume \(w \equiv w'\). Then there exist positive words \(v, v'\) satisfying \(N(w) v \equiv^+ N(w') v'\) and \(D(w) v \equiv^+ D(w') v'\). By definition of the ‘dil’ mapping and Lemma VII.4.5, we have

\[
\text{dil}(\text{dil}(1, D(w)), v) = \text{dil}(1, D(w)v) = \text{dil}(1, D(w')v') = \text{dil}(\text{dil}(1, D(w')), v'), \\
\text{dil}(\text{dil}(1, N(w)), v) = \text{dil}(1, N(w)v) = \text{dil}(1, N(w')v') = \text{dil}(\text{dil}(1, N(w')), v').
\]

By construction, the mappings \(\text{dil}(\cdot, v)\) and \(\text{dil}(\cdot, v')\) are increasing, so we deduce that \(\text{dil}(1, D(w)) < \text{dil}(1, N(w))\) is equivalent to \(\text{dil}(1, D(w')) < \text{dil}(1, N(w'))\), and the same for = and >.

Assume now \(w_1, w_2 \in P_+\). We find

\[
\text{dil}(1, D(w_1 w_2)) = \text{dil}(1, D(w_2) (N(w_2) \setminus D(w_1))) = \text{dil}(1, D(w_2)), N(w_2) \setminus D(w_1)) < \text{dil}(\text{dil}(1, D(w_2)), N(w_2) \setminus D(w_1)) = \text{dil}(1, D(w_1) (D(w_1) \setminus N(w_2))) = \text{dil}(1, D(w_1)) \setminus N(w_2)) < \text{dil}(1, D(w_1)), D(w_1) \setminus N(w_2)) = \text{dil}(1, N(w_1) (D(w_1) \setminus N(w_2))) = \text{dil}(1, N(w_1) (D(w_1) \setminus N(w_2))) = \text{dil}(1, N(w_1)).
\]

The computation is similar in the other cases. \[\blacksquare\]

**Definition 3.3.** (relations \(\prec\) and \(\simeq\)) For \(a, b\) in \(G_{LD}\), we say that \(a \prec b\) (resp. \(a \simeq b\)) holds if \(a^{-1}b\) admits an expression in \(P_+\) (resp. in \(P_0\)).

Applying Lemma 3.2, we obtain directly:

**Proposition 3.4.** (preorder) The relation \(\prec\) is a preorder on \(G_{LD}\), and \(\simeq\) is the associated equivalence relation; both are invariant under left multiplication.

We deduce a new solution of the word problem for \((LD)\) in the case of one variable terms:
Proposition 3.5. (word problem) For all terms $t_1$, $t_2$ in $T_1$, $t_1 \sqsubseteq_{ld} t_2$ is equivalent to $\chi_{t_1} \prec \chi_{t_2}$, and $t_1 \equiv_{ld} t_2$ is equivalent to $\chi_{t_1} \simeq \chi_{t_2}$.

Proof. The computation at the beginning of this section shows that $\chi_{t_1}^{-1} \cdot \chi_{t_2}$ belonging to $P_\varphi$, $P_0$, or $P_\sigma$ implies $t_1 \sqsubseteq_{ld} t_2$, $t_1 \equiv_{ld} t_2$, or $t_1 \simeq_{ld} t_2$ respectively. The latter cases exclude each other, so the implication is an equivalence.

Example 3.6. (word problem) Every factor in the above expressions is effectively computable. For instance, let $t_1 = (x \cdot x) \cdot ((x \cdot x))$ and $t_2 = x \cdot ((x \cdot x) \cdot (x \cdot x))$. We compute

\[
\chi_{t_1} = \phi \cdot 11 \cdot 1 \cdot \phi \cdot 1^{-1}, \quad \chi_{t_2} = 1 \cdot 11 \cdot 1 \cdot 11^{-1} \cdot \phi,
\]

\[
D(\chi_{t_1}^{-1} \cdot \chi_{t_2}) = 11 \cdot 1 \cdot 0 \cdot 11 \cdot 01 \cdot 10 \cdot 00, \quad N(\chi_{t_1}^{-1} \cdot \chi_{t_2}) = 1 \cdot 11 \cdot 01 \cdot 1 \cdot 0,
\]

and, finally, $\rho(1, D(\chi_{t_1}^{-1} \cdot \chi_{t_2})) = 1 = \rho(1, N(\chi_{t_1}^{-1} \cdot \chi_{t_2}))$. So we have $\chi_{t_1}^{-1} \cdot \chi_{t_2} \in P_0$, hence $t_1 \equiv_{ld} t_2$.

Another consequence is:

Corollary 3.7. Assume that $a$, $b$ are special elements of $G_{ld}$. Then

(i) $a \prec b$ holds in $G_{ld}$ if and only if $pr(a) \prec_{l} pr(b)$ holds in $B_\infty$;

(ii) $a \simeq b$ holds in $G_{ld}$ if and only if $pr(a) = pr(b)$ holds in $B_\infty$.

Proof. Assume $a = \chi_{t_1}$, $b = \chi_{t_2}$. By Proposition 3.4, $a \prec b$ is equivalent to $t_1 \sqsubseteq_{ld} t_2$, which, by Proposition III.2.14(ii) (order), is equivalent to $t_1(1)^{B_{\infty}} \prec_{l} t_2(1)^{B_{\infty}}$. Now, by construction, $t(1)$ evaluated in $(B_{\infty}, \prec)$ is $pr(\chi_{t_1})$ for every term $t$ in $T_1$. The argument is similar for $\simeq$.

A natural question is whether the equivalences of Corollary 3.7, which connect the order on special braids with the preorder $\prec$ on $G_{ld}$, extend to the whole of $G_{ld}$. They do not, for the definition of $\prec$ takes into account the image of the left subterm only. For instance, we have $\rho(1, \phi \cdot 0) = \rho(1, \phi \cdot \phi) = 3$, hence $g_0 \simeq g_0^2$ holds in $G_{ld}$, while $pr(g_0 \simeq g_0^2) = \sigma_1 \neq pr(g_0^2) = \sigma_1^2$ holds in $B_\infty$.

The last point in the proof of Proposition 3.4 (preorder), namely the relations $t_1 \sqsubseteq_{ld} t_2$, $t_1 \equiv_{ld} t_2$, or $t_1 \simeq_{ld} t_2$ excluding each other, is equivalent to the statement that $t \sqsubseteq_{ld} t$ holds for no term $t$ in $T_1$, i.e., to the property that left division in the free LD-system $FLD_1$ has no cycle. This property has been deduced in Section I.3 from the similar property for exponentiation on $Aut(FG_\infty)$. With the current tools, we can give a purely syntactic—and very simple—proof.

Proposition 3.8. (syntactic acyclicity) No term $t$ in $T_1$ satisfies $t \sqsubseteq_{ld} t$. 

Proof. It suffices to prove directly that \( t_1 \sqsubseteq_{LD} t_2 \) implies \( \chi_{t_1}^{-1} \cdot \chi_{t_2} \in P_+ \). Then \( t \sqsubseteq_{LD} t \) would imply \( \varepsilon \in P_+ \), which is impossible, as \( P_+ \) and \( P_0 \) are disjoint. So, assume \( t_1 \sqsubseteq_{LD} t_2 \). By definition, there exist terms \( t'_1, t'_2 \) satisfying \( t'_1 \sqsubseteq_{LD} t_1, \ t'_2 \sqsubseteq_{LD} t_2 \) and \( t'_1 \) is an iterated left subterm of \( t'_2 \), say \( t'_1 = \text{left}^p(t'_2) \). By Proposition 2.8 (action I), there exist two words \( w_1, w_2 \) such that \( \chi_{t_e} \equiv \chi_{t'_1} \cdot 0w_e \) holds for \( e = 1, 2 \), and, therefore, we have

\[
\chi_{t_1}^{-1} \cdot \chi_{t_2} \equiv 0w_1^{-1} \cdot \chi_{t'_1}^{-1} \cdot \chi_{t'_2} \cdot 0w_2. \tag{3.2}
\]

Assume \( t'_2 = ((t'_1 \cdot s_1) \cdot s_2) \cdots s_p \). Applying the definition of the blueprint gives a decomposition of the form

\[
\chi_{t'_2} = \chi_{t'_1} \cdot 1u_0 \cdot \phi \cdot 1u_1 \cdot \phi \cdots \phi \cdot 1u_p,
\]

and we deduce from (3.2) that \( \chi_{t_1}^{-1} \cdot \chi_{t_2} \) is equivalent to

\[
0w_1^{-1} \cdot 1u_0 \cdot \phi \cdot 1u_1 \cdot \phi \cdots \phi \cdot 1u_p \cdot 0w_2. \tag{3.3}
\]

Now, \( \phi \) belongs to \( P_+ \), as we have \( \text{dil}(1, \phi) = 2 \), and \( \text{dil}(1, \varepsilon) = 1 \). On the other hand, every word of the form \( 0u \) or \( 1u \) belongs to \( P_0 \) by definition. Hence, the word in (3.3) belongs to \( P_+ \), and so does \( \chi_{t_1}^{-1} \cdot \chi_{t_2} \).

The left ordering of terms

Terms can be equipped with several orderings in connection with their tree structure. Here, we introduce a linear order on \( T_\infty \) that uses the left height as a discriminant.

Terms have been constructed as words, and we have used the right Polish form where the product of \( t_1 \) and \( t_2 \) is defined to be the word \( t_1 t_2 \). Here we shall resort for a while to the left Polish form, where the product of \( t_1 \) and \( t_2 \) is defined to be \( \bullet t_1 t_2 \).

Definition 3.9. (left Polish form) For \( t \) a term, the left Polish form of \( t \) is the word \( (t)_{\ell} \) inductively defined by \( (t)_{\ell} = t \) for \( t \) a variable, and \( (t_1 t_2)_{\ell} = \bullet (t_1)_{\ell} (t_2)_{\ell} \).

For instance, we have

\[
(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)) \cdot x_5)_{\ell} = (x_1 x_2 x_3 x_4 \bullet x_5 \bullet)_{\ell} = \bullet x_1 \bullet x_2 \bullet x_3 x_4 x_5.
\]

When the term \( t \) is viewed as a tree, the word \( (t)_{\ell} \) is obtained by enumerating the variables of \( t \) from left to right and writing before the variable occurring at \( \alpha \) as many letters \( \bullet \) as there are final 0’s in \( \alpha \), a counterpart to right Polish form, where the variable occurring at \( \alpha \) is followed by as many letters \( \bullet \) as there are final 1’s in \( \alpha \).
Definition 3.10. (left order) For \( t_1, t_2 \in T_\infty \), we say that \( t_1 \preceq_L t_2 \) holds if the word \((t_1)_L\) precedes the word \((t_2)_L\) in the lexicographical extension of \( x_1 < x_2 < \ldots < \bullet \).

By construction, the relation \( \preceq_L \) is a linear order on \( T_\infty \), and \( x_1 \) is minimal for \( \preceq_L \). Letting \( h_L(t) \) denote the left height of the term \( t \), i.e., the length of the leftmost branch of \( t \), we see that the word \((t)_L\) begins with \( h_L(t) \) letters \( \bullet \) followed by a variable. So, \( h_L(t_1) < h_L(t_2) \) implies \( t_1 <_L t_2 \). Observe that the linear order \( <_L \) does not include the partial orders connected with the right Polish form considered in Chapter V: for instance, we have \( x_1 \cdot x_2 \prec x_2 \), hence \( x_1 \cdot x_2 < x_2 \), but \( x_1 \cdot x_2 >_L x_2 \) holds, as the word \((x_1 \cdot x_2)_L\), namely \( \bullet x_1 x_2 \), lies after \( x_2 \).

Our aim is to prove that the action of \( G_{LD} \) on \( T_\infty \) preserves \( <_L \):

Proposition 3.11. (invariance) Assume \( t, t' \in T_\infty \). Then \( t <_L t' \) is equivalent to \( t \cdot a <_L t' \cdot a \) for every \( a \) in \( G_{LD} \) such that \( t \cdot a \) and \( t' \cdot a \) exist (if any).

The proof begins with auxiliary results about the left Polish notation and the left ordering of terms. First, we have the following counterpart to Lemma V.1.7 (we recall that \( \#_w(w) \) and \( \#_L(w) \) denote the number of letters from \( X \) and of \( \bullet \) in \( w \) respectively).

Lemma 3.12. A word \( w \) on the alphabet \( X \cup \{\bullet\} \) is the left Polish form of a term if and only if we have \( \#_w(w) = \#_L(w) + 1 \) and \( \#_w(u) \leq \#_L(u) \) for every proper prefix \( u \) of \( w \).

Lemma 3.13. The inequality \( t_1 <_L t_2 \) implies \( t_1 t_3 <_L t_2 t_4 \) for all \( t_3, t_4 \).

Proof. Lemma 3.12 implies that a proper prefix of the left Polish form of a term is never the left Polish form of a term. Hence \( t_1 <_L t_2 \) holds if and only if the words \((t_1)_L\) and \((t_2)_L\) have a clash of the type “variable vs. \( \bullet \)”. Then the words \((t_1)_L\) and \((t_2)_L\), i.e., \( (t_1)_L(t_3)_L \) and \( (t_2)_L(t_4)_L \), have a similar clash.

Lemma 3.14. Assume \( t_1, t_2 \in T_\infty \). If \( t_1 \) is a variable, say \( t_1 = x_i \), \( t_1 <_L t_2 \) holds unless we have \( t_2 = x_j \) with \( j < i \). If \( t_1 \) is not a variable, then \( t_1 <_L t_2 \) holds if and only if we have either \( \text{sub}(t_1, 0) <_L \text{sub}(t_2, 0) \), or \( \text{sub}(t_1, 0) = \text{sub}(t_2, 0) \) and \( \text{sub}(t_1, 1) <_L \text{sub}(t_2, 1) \).

Proof. Assume \( t_1 <_L t_2 \), and neither \( t_1 \) nor \( t_2 \) are variables. Three cases are possible. If \( \text{sub}(t_1, 0) <_L \text{sub}(t_2, 0) \), then, by Lemma 3.13, \( t_1 <_L t_2 \) holds. Symmetrically, if \( \text{sub}(t_1, 0) >_L \text{sub}(t_2, 0) \) holds, then \( t_1 >_L t_2 \) holds. Finally, if \( \text{sub}(t_1, 0) \) and \( \text{sub}(t_2, 0) \) are equal, then, by definition, \( t_1 <_L t_2 \) holds if and only if \( \text{sub}(t_1, 1) <_L \text{sub}(t_2, 1) \) does.
Lemma 3.15. Assume that \( t_1, t_2 \) are terms in \( T_\infty \). Then the following are equivalent:

(i) The relation \( t_1 \preceq t_2 \) holds;

(ii) There exists an address \( \alpha \) in \( \text{Skel}(t_1) \cap \text{Skel}(t_2) \) such that \( \text{sub}(t_1, \beta) = \text{sub}(t_2, \beta) \) holds for every \( \beta \) in the left edge of \( \alpha \), and \( \text{sub}(t_1, \alpha) \preceq \text{sub}(t_2, \alpha) \) holds;

(iii) There exists an address \( \alpha \) in \( \text{Out}(t_1) \cap \text{Skel}(t_2) \) such that \( \text{sub}(t_1, \beta) = \text{sub}(t_2, \beta) \) holds for every \( \beta \) in the left edge of \( \alpha \), and either \( \text{sub}(t_2, \alpha) \) is a variable larger than \( \text{var}(t_1, \alpha) \), or it is not a variable.

Proof. An induction on \( \alpha \) shows that, if \( \alpha \) belongs to the skeleton of the term \( t \) and \( (\alpha_1, \ldots, \alpha_p) \) is the left edge of \( \alpha \), then the word \( (t)_L \) begins with

\[
\bullet^{k_2}(\text{sub}(t, \alpha_1))_L \bullet^{k_2}(\text{sub}(t, \alpha_2))_L \ldots \bullet^{k_p}(\text{sub}(t, \alpha_p))_L \bullet^k(\text{sub}(t, \alpha))_L, \tag{3.4}
\]

where \( k_i \) is the number of final 0’s in \( \alpha_i \), and \( k \) is the number of final 0’s in \( \alpha \); the result is obvious for \( \alpha = \emptyset \), and, otherwise, we apply the inductive definition of the left edge as given in Chapter VI. Then, by definition of a lexicographical ordering, it follows from (3.4) and from the fact that a proper prefix of a left Polish form is never the left Polish form of a term that (ii) implies (i).

By construction, (iii) implies (ii). Finally, assuming (i), and letting \( \alpha \) be the address of the first position where the words \( (t_1)_L \) and \( (t_2)_L \) disagree, we obtain (iii) using the explicit expansion of (3.4).

Lemma 3.16. Assume that \( t_1, t_2 \) are \( \preceq_{LD} \)-comparable terms in \( T_\infty \). Then the following are equivalent:

(i) The relation \( t_1 \preceq_{LD} t_2 \) holds;

(ii) There exists an address \( \alpha \) in \( \text{Out}(t_1) \cap \text{Skel}(t_2) \) such that \( \text{sub}(t_1, \beta) = \text{sub}(t_2, \beta) \) holds for every \( \beta \) in the left edge of \( \alpha \), and \( \text{sub}(t_2, \alpha) \) is a term of left height at least 1 whose leftmost variable is \( \text{var}(t_1, \alpha) \).

Proof. Assume (i). Then there exists an address \( \alpha \) satisfying the conditions of Lemma 3.15(iii). We claim that \( \text{sub}(t_2, \alpha) \) cannot be a variable. Indeed, assume \( \text{sub}(t_1, \alpha) = x_i \) and \( \text{sub}(t_2, \alpha) = x_j \) with \( j > i \). Then, letting \( (\alpha_1, \ldots, \alpha_p) \) denote the left edge of \( \alpha \), we see that (the right Polish form of) \( t_1 \) begins with \( \text{sub}(t_1, \alpha_1) \ldots \text{sub}(t_1, \alpha_p)x_i \), while (the right Polish form of) \( t_2 \) begins with \( \text{sub}(t_1, \alpha_1) \ldots \text{sub}(t_1, \alpha_p)x_j \). Thus \( t_1 \preceq t_2 \) holds, contradicting the hypothesis that \( t_1 \) and \( t_2 \) are \( \preceq_{LD} \)-comparable. So the only possibility is \( \text{sub}(t_2, \alpha) \) not to be a variable, and its leftmost variable to be \( x_i \). This gives (ii). Moreover, the conditions of Lemma 3.15(iii) hold for the terms \( t_1^1 \) and \( t_2^1 \) as well, so we have \( t_1^1 \preceq_{LD} t_2^1 \), and (iii) holds as well. That (ii) implies (i) follows from Lemma 3.15.

We deduce that the left order of terms is invariant under substitution:
Proposition 3.17. (substitution) Assume that \( t_1, t_2 \) belong to \( T_\infty \), \( f \) is a substitution of \( T_\infty \), and at least one of the following conditions holds:

(i) We have \( f(x_i) <_t f(x_{i+1}) \) and \( \text{ht}_t(f(x_i)) = \text{ht}_t(f(x_{i+1})) \) for every \( i \);

(ii) The terms \( t_1 \) and \( t_2 \) are \( \subseteq_{\text{ld}} \)-comparable.

Then \( t_1 <_t t_2 \) is equivalent to \( t'_1 <_t t'_2 \).

Proof. As \( <_t \) is a linear order, it suffices to prove that \( t_1 <_t t_2 \) implies \( t'_1 <_t t'_2 \).

So assume \( t_1 <_t t_2 \). By Lemma 3.15, there exists an address \( \alpha \) such that \( \text{sub}(t_1, \beta) = \text{sub}(t_2, \beta) \) holds for every \( \beta \) in the left edge of \( \alpha \), \( \text{sub}(t_1, \alpha) \) is a variable say \( x_i \), and \( \text{sub}(t_2, \alpha) \) is either a variable \( x_j \) with \( j > i \), or it is a term that is not a variable. When Condition (ii) holds, by Lemma 3.16, we can assume that \( \text{sub}(t_2, \alpha) \) is a term with leftmost variable \( x_i \) and left height at least 1. Applying \( f \), we deduce \( \text{sub}(t'_1, \beta) = \text{sub}(t'_2, \beta) \) for every \( \beta \) in the left edge of \( \alpha \). Then we have \( \text{sub}(t'_1, \alpha) = f(x_i) \). We claim that \( \text{sub}(t'_1, \alpha) \not<_{_t} \text{sub}(t'_2, \alpha) \) always holds, which, by Lemma 3.15, implies \( t'_1 <_t t'_2 \). Three cases are to be considered.

If Condition (i) holds and \( \text{sub}(t_2, \alpha) \) is \( x_j \) with \( j > i \), we obtain \( \text{sub}(t'_1, \alpha) = f(x_i) <_{_t} f(x_j) = \text{sub}(t'_2, \alpha) \). If Condition (i) holds and \( \text{sub}(t_2, \alpha) \) is not a variable, the hypothesis on \( f \) implies \( \text{ht}_t(\text{sub}(t'_2, \alpha)) > \text{ht}_t(\text{sub}(t'_1, \alpha)) \), hence \( \text{sub}(t'_1, \alpha) \not<_{_t} \text{sub}(t'_2, \alpha) \). If Condition (ii) holds and \( \text{sub}(t_2, \alpha) \) is a term with leftmost variable \( x_i \) and left height \( k \geq 1 \), we find \( \text{ht}_t(\text{sub}(t'_1, \alpha)) = \text{ht}_t(f(x_i)) \), and \( \text{ht}_t(\text{sub}(t'_2, \alpha)) = \text{ht}_t(f(x_i)) + k \), hence \( \text{sub}(t'_1, \alpha) \not<_{_t} \text{sub}(t'_2, \alpha) \).

The previous result implies that every substitution of \( T_1 \) preserves \( <_{_t} \), and that, if \( t_1 \) and \( t_2 \) are \( \subseteq_{\text{ld}} \)-comparable terms in \( T_\infty \), then \( t_1 <_t t_2 \) is equivalent to \( t'_1 <_t t'_2 \), where we recall \( t' \) denotes the projection of \( t \) on \( T_1 \).

We are now ready to prove Proposition 3.11 (invariance).

Proof. As \( G_{\text{ld}} \) is generated by the elements \( g_\alpha \) with \( \alpha \in A \), it suffices to prove the result for the latter elements, i.e., to prove that, if \( \alpha \) is an address, and \( t, t' \) are terms in the domain of the operator \( \text{ld}_\alpha \), then \( t <_{_t} t' \) is equivalent to \( t \cdot \alpha <_{_t} t' \cdot \alpha \). As the action of \( \alpha \) is injective, it suffices to prove that \( t <_t t' \) implies \( t \cdot \alpha <_{_t} t' \cdot \alpha \). We use induction on \( \alpha \). Assume first \( \alpha = \emptyset \). The hypothesis that \( t \) and \( t' \) belong to the domain of \( \text{ld}_\emptyset \) implies that we can write \( t = t_1 \cdot (t_2 \cdot t_3), t' = t'_1 \cdot (t'_2 \cdot t'_3) \), and we have the explicit decompositions

\[
(t) = (t_1)(t_2)(t_3) \quad \text{and} \quad (t \cdot \emptyset) = (t_1)(t_2),
\]

and similar dashed counterparts. By Lemma 3.14, three cases only are possible, namely \( t_1 <_{_t} t'_1 \), or \( t_1 = t'_1 \) and \( t_2 <_{_t} t'_2 \), or \( t_1 = t'_1 \), \( t_2 = t'_2 \) and \( t_3 <_{_t} t'_3 \); the result is clear in each case.

Assume now \( \alpha = 0 \beta \). Write \( t = t_1 \cdot t_2 \) and \( t' = t'_1 \cdot t'_2 \). Then we have \( t \cdot \alpha = (t_1 \cdot \beta) t_2 \) and \( t' \cdot \alpha = (t'_1 \cdot \beta) t'_2 \). Two cases are possible. For \( t_1 <_{_t} t'_1 \), the
induction hypothesis implies \( t_1 \cdot \beta \leq t_1' \cdot \beta \), and, therefore, \( t \cdot \alpha \leq t' \cdot \alpha \). For \( t_1 = t_1' \) and \( t_2 \leq t_2' \), we have \( t_1 \cdot \beta = t_1' \cdot \beta \), and, again, \( t \cdot \alpha \leq t' \cdot \alpha \).

Assume finally \( \alpha = 1. \beta \). Then we have \( t \cdot \alpha = t_1 \cdot (t_2 \cdot \beta) \) and \( t' \cdot \alpha = t_1' \cdot (t_2' \cdot \beta) \). Two cases are possible again. For \( t_1 < t_1' \), we obtain \( t \cdot \alpha < t' \cdot \alpha \) directly. For \( t_1 = t_1' \) and \( t_2 \leq t_2' \), the induction hypothesis implies \( t_2 \cdot \beta \leq t_2' \cdot \beta \), and we deduce \( t \cdot \alpha \leq t' \cdot \alpha \) again. \( \blacksquare \)

A linear order on \( G_{LD} \)

In Chapter III, we used the action of braids on the linearly ordered set \((B_{sp}, \subseteq)^{\mathbb{N}}\) to define a linear order on \( B_{sp} \). Here we use the action of \( G_{LD} \) on the linearly ordered set \((T_{\omega}, \subseteq)\) to similarly define a linear order on \( G_{LD} \). The order so defined is compatible with multiplication on both sides, so \( G_{LD} \) is a bi-orderable group.

We begin with an ordering on the submonoid \( G_{LD}^{+} \) of \( G_{LD} \). We recall from Chapter VII that, for all words \( u, v \) on \( A \), there exists a unique canonical term \( t_{u,v}^L \) such that the intersection of the domains of \( \text{LD}_u \) and \( \text{LD}_v \) is the set of substitutes of \( t_{u,v}^L \). As \( u \equiv u' \) and \( v \equiv v' \) imply \( t_{u,v}^L = t_{u',v'}^L \), \( t^L \) induces a well defined mapping, still denoted \( t^L \), on \( G_{LD}^+ \times G_{LD}^+ \).

**Lemma 3.18.** For \( a, b \in G_{LD}^{+} \), the following are equivalent:

(i) There exists a term \( t \) in \( T_{\omega} \) such that \( t \cdot a \leq t \cdot b \) holds;

(ii) The inequality \( t_{a,b}^L \cdot a < t_{a,b}^L \cdot b \) holds;

(iii) For every term \( t \) such that \( t \cdot a \) and \( t \cdot b \) exist, \( t \cdot a < t \cdot b \) holds.

**Proof.** That (ii) implies (i) and (iii) implies (ii) is clear. So assume (i). By construction, there exists a substitution \( f \) satisfying \( t = (t_{a,b}^L)^f \), and our hypothesis is the inequality \((t_{a,b}^L)^f \cdot a < (t_{a,b}^L)^f \cdot b, i.e., (t_{a,b}^L \cdot a)^f \leq (t_{a,b}^L \cdot b)^f \). The terms \( t_{a,b}^L \cdot a \) and \( t_{a,b}^L \cdot b \) are LD-equivalent, hence, by Proposition 3.17 (substitution), the previous inequality is equivalent to \( t_{a,b}^L \cdot a < t_{a,b}^L \cdot b \), which gives (ii), and, then, to \((t_{a,b}^L \cdot a)^g < (t_{a,b}^L \cdot b)^g \) for every substitution \( g \), which gives (iii). \( \blacksquare \)

**Definition 3.19.** (Order on \( G_{LD}^{+} \)) For \( a, b \in G_{LD}^{+} \), we say that \( a < b \) holds if the equivalent conditions of Lemma 3.18 are satisfied.

**Proposition 3.20.** (Order on \( G_{LD}^{+} \)) (i) The relation \( < \) is a linear order on the monoid \( G_{LD}^{+} \) that is both left and right compatible with multiplication, and admits 1 as a minimal element.

(ii) If \( a, b, a', b' \) are elements of \( G_{LD}^{+} \) satisfying \( ab^{-1} = a'b'^{-1} \), then \( a < b \) is equivalent to \( a' < b' \).
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Proof. (i) The relation $<$ is irreflexive as $<_L$ is. Assume $a < b < c$. There exist two substitutions $f, g$ satisfying

$$t_{a,b,c}^b = (t_{a,b}^b)^f = (t_{b,c}^f)^g. \tag{3.5}$$

By Lemma 3.18, $a < b$ implies $t_{a,b}^b < t_{a,b,c}^b$, hence, by (3.5), $t_{a,b,c}^b < t_{a,b,c}^b$. Similarly, $b < c$ implies $t_{b,c}^f < t_{b,c}^c$, hence, by (3.5), $t_{a,b,c}^b < t_{a,b,c}^c$. By transitivity of $<_L$, we deduce $t_{a,b,c}^b < t_{a,b,c}^c$, which in turn gives $a < c$ by Lemma 3.18.

Assume now $a < b$, and let $c$ be an arbitrary element of $G_{LD}^+$. Let $t$ be a term in the domains of $LD_{ca}$ and $LD_{cb}$. By construction, the term $t \cdot c$ belongs to the domain of $LD_a$ and $LD_b$, so the hypothesis $a < b$ implies $(t \cdot c) \cdot a < (t \cdot c) \cdot b$, which in turn implies $a \cdot c < b \cdot c$ by definition. With the same hypotheses, assume that $t$ belongs to the domains of $LD_{ac}$ and $LD_{bc}$. Then $t \cdot a < t \cdot b$ holds by hypothesis, and, by Proposition 3.11 (invariance under $G_{LD}$), this implies $(t \cdot a) \cdot c < (t \cdot b) \cdot c$, hence $ac < bc$ by definition.

Finally, assume $a \neq 1$. We claim that $t <_L t \cdot a$ holds whenever $t \cdot a$ is defined. It suffices to consider the case of a single address $\alpha$. For $\alpha = \emptyset$, the result follows from the equality $ht_c(t \cdot \emptyset) = ht_c(t) + 1$. Otherwise, we use induction on $\alpha$, or we resort to Lemma 3.15: by the previous argument, we have $\operatorname{sub}(t, \alpha) <_L \operatorname{sub}(t \cdot \alpha, \alpha)$, and, by construction, $\operatorname{sub}(t, \beta) = \operatorname{sub}(t \cdot \alpha, \beta)$ holds for every $\beta$ in the left edge of $\alpha$.

(ii) By Proposition 1.14 (fraction), there exist $c, c'$ in $G_{LD}$ satisfying $ac = a'c'$ and $bc = b'c'$. Assume $a < b$. Using the compatibility of the order with multiplication, we deduce $ac < bc$, i.e., $a'c' < b'c'$, hence $a' \leq b'$. \hfill \ensuremath{\blacksquare}

Definition 3.21. (order on $G_{LD}$) For $c, d$ in $G_{LD}$, we say that $c < d$ holds if $cd^{-1} = ab^{-1}$ holds for some elements $a, b$ of $G_{LD}^+$ satisfying $a < b$.

Proposition 3.22. (order on $G_{LD}$) (i) The relation $<$ is a linear order on $G_{LD}$ that extends the order $<_L$ on $G_{LD}^+$. This order is compatible with multiplication on both sides, and, therefore, it is compatible with conjugacy.

(ii) The group $G_{LD}$ is a bi-orderable group. Hence $G_{LD}$ is torsion free, and the group algebra $CG_{LD}$ admits no zero divisor.

Proof. (i) For $a, b$ in $G_{LD}^+$, $1 = ab^{-1}$ implies $a = b$, hence $a \notin b$, so, for every $c$ in $G_{LD}$, $c < c$ is impossible. Assume now $c_1 < c_2 < c_3$ in $G_{LD}$. There exist $a_1, b_1, a_2, b_2$ in $G_{LD}^+$ satisfying $c_1c_2^{-1} = a_1b_1^{-1}$, $c_2c_3^{-1} = a_2b_2^{-1}$, $a_1 < b_1$, and $a_2 < b_2$. Let $a_3, b_3$ be elements of $G_{LD}^+$ satisfying $a_2b_3 = a_1b_3$. We find

$$c_1c_3^{-1} = a_1b_1^{-1}a_2b_2^{-1} = (a_1a_3)(b_2b_3)^{-1}.\]$$

Then $a_1 < b_1$ implies $a_1a_3 < b_1a_3$, and $a_2 < b_2$ implies $a_2b_3 < b_2b_3$. By hypothesis, we have $b_1a_3 = a_2b_3$, so we deduce $a_1a_3 < b_2b_3$, and, therefore, $c_1 < c_3$. Hence $<$ is an ordering on $G_{LD}$. 


Assume that \( a, b \) belong to \( G_{LD}^+ \) and \( a < b \) holds in the sense of \( G_{LD}^+ \). Then \( ab^{-1} \) is an expression of \( ab^{-1} \) with \( a, b \) in \( G_{LD}^+ \) and \( a < b \), and \( a < b \) holds in the sense of \( G_{LD} \). Thus \(< \) on \( G_{LD} \) extends \(< \) on \( G_{LD}^+ \).

Assume now \( c_1, c_2, c \in G_{LD} \) and \( c_1 < c_2 \). By definition, there exist \( a, b \) in \( G_{LD}^+ \) satisfying \( c_1 = c_2^{-1} = ab^{-1} \) and \( a < b \). Then we have \((c_1c)(c_2c)^{-1} = ab^{-1}\), so \( c_1c < c_2c \) holds as well. On the other hand, let us express \( c \) as \( a_0b_0^{-1} \) with \( a_0, b_0 \) in \( G_{LD}^+ \). There exist \( a_1, b_1, a_2, b_2, a_3, b_3 \) in \( G_{LD}^+ \) satisfying \( b_0a_1 = a_2b_2 \), \( b_0b_1 = b_2b_2 \), \( a_2a_3 = b_2b_3 \) (Figure 4.1), and we find

\[
(cc_1)(cc_2)^{-1} = a_0b_0^{-1}ab^{-1}b_0a_0^{-1} = (a_0a_1a_3)(a_0b_1b_3)^{-1}.
\]

(3.6)

Then \( a < b \) implies \( b_0a_1a_3 = a_2a_3 < b_0a_2a_3 = b_2b_3 = b_0b_1b_3 \), from which we deduce \( a_1a_3 < b_1b_3 \), and, therefore, \( a_0a_1a_3 < a_0b_1b_3 \) using compatibility with multiplication on the left twice. By (3.6), this gives \( cc_1 < cc_2 \).

Point (ii) is a straightforward consequence of (i).

Figure 4.1. Compatibility with left multiplication

The action of the group \( G_{LD} \) on terms is a partial action. In particular, some elements \( c \) of \( G_{LD} \) do not act, i.e., the operator LD is empty. However, when the action exists, the connection between the order in \( G_{LD} \) and the left order of terms is what we can expect:

**Proposition 3.23.** (Action) (i) For \( c, d \in G_{LD} \) and \( t \) in \( T_\infty \) such that \( t \cdot c \) and \( t \cdot d \) exist, \( c < d \) is equivalent to \( t \cdot c < t \cdot d \).

(ii) For \( c \in G_{LD} \) and \( t \) in \( T_\infty \) such that \( t \cdot c \) exists, \( c > 1 \) is equivalent to \( t \cdot c >_L t \).

**Proof.** (i) Assume that \( w_1, w_2 \) are words on \( A \cup A^{-1} \) that represent \( c \) and \( d \) respectively, and the terms \( t \cdot w_1 \) and \( t \cdot w_2 \) are defined. Let \( t' = t \cdot w_1 \). Then \( t' \cdot w_1^{-1}w_2 \) is defined. By Proposition 1.11 (convergence), there exist two words \( u, v \) on \( A \) such that \( w_1^{-1}w_2 \) is reversible to \( uv^{-1} \). Then, by Lemma 3.23, the term \( t' \cdot uv^{-1} \) exists, and we have \( t' \cdot uv^{-1} = t' \cdot w_1^{-1}w_2 = t \cdot w_2 = t \cdot d \). Let \( a \) and \( b \) be the classes of \( u \) and \( v \) in \( G_{LD}^+ \). By construction, we have \( c^{-1}d = ab^{-1} \).

Assume \( c < d \). By construction, we have \( ca = db \), hence \( a > b \). The term \( t \cdot d \) need not belong to the domain of the operator LD in general. However, as \( u \)
is a positive word, the only possible obstruction is the skeleton of \( t \) being too small. Hence, at the expense of possibly replacing \( t \) with a sufficiently large substitute, we may assume that \( t \cdot da \) is defined. Now the hypothesis \( a > b \) implies \( t \cdot da > L t \cdot db = t \cdot ca \), and, therefore, \( t \cdot d > L t \cdot c \). The argument is symmetric for \( c > d \).

Point (ii) follows from (i) by taking \( d = 1 \). \hfill \blacksquare

**Corollary 3.24.** For \( t_1, t_2 \) in \( T_1 \), we have \( t_1 < L t_2 \) in \( T_1 \) if and only if \( x_{t_1} < x_{t_2} \) holds in \( G_{LD} \).

**Proof.** By construction, we have

\[
x^{[p+1]} \cdot x_{t_1} = t_1 \cdot x^{[p]} \quad \text{and} \quad x^{[p+1]} \cdot x_{t_2} = t_2 \cdot x^{[p]}
\]

for \( p \) large enough. By Proposition 3.23 (action), the inequality \( x_{t_1} < x_{t_2} \) holds in \( G_{LD} \) if and only if the inequality \( t_1 \cdot x^{[p]} < L t_2 \cdot x^{[p]} \) holds in \( T_\infty \), hence in \( T_1 \). The latter is equivalent to \( t_1 < L t_2 \). \hfill \blacksquare

**Exercise 3.25. (term order)** (i) Assume that \( t, t_1, t_2 \) are terms in \( T_\infty \) and the outline of \( t \) is included in the skeleton of \( t_1 \) and \( t_2 \). Let \( (a_1, \ldots, a_p) \) be the left–right enumeration of \( \text{Out}(t) \). Prove that \( t_1 < L t_2 \) holds if and only if the sequence \( \text{sub}(t_1, a_1), \ldots, \text{sub}(t_1, a_p) \) precedes the sequence \( \text{sub}(t_2, a_1), \ldots, \text{sub}(t_2, a_p) \) in the lexicographical extension of \( <_L \) to \( T_\infty \).

(ii) Assume that \( t_1 \) and \( t_2 \) belong to \( T_1 \). Prove that \( t_1 < L t_2 \) holds if and only if \( \text{Out}(t_1) \sqsubseteq^* \text{Out}(t_2) \) does, where \( \text{Out}(t) \) denotes the left–right increasing enumeration of the outline of \( t \), and \( \sqsubseteq^* \) denotes the lexicographical extension to \( A^* \) of the prefix ordering \( \sqsubseteq \) on \( A \).

**Exercise 3.26. (right ordering)** Denote by \( <_{RH} \) the linear order on \( T_\infty \) obtained by lexicographically extending \( \bullet < x_1 < x_2 < \ldots \) to (right Polish) terms. Investigate the compatibility of \( <_{RH} \) with substitutions and action of \( G_{LD} \). Deduce a new linear order on \( G_{LD}^+ \) which is compatible with left multiplication. Is it compatible with right multiplication?

**Exercise 3.27. (counter-examples)** (i) Find two terms \( t_1, t_2 \) satisfying \( t_1 \sqsubseteq_{LD} t_2 \) and \( t_1 >_{LD} t_2 \). Deduce two elements \( c, d \) in \( G_{LD} \) satisfying \( c < d \) and \( c > d \). [Hint: Take \( c = \chi_{t_1} \) and \( d = \chi_{t_2} \).]

(ii) Find two elements \( c, d \) in \( G_{LD} \) such that \( c < d \) holds in \( G_{LD} \), while \( \text{pr}(c) >_{L} \text{pr}(d) \) holds in \( B_\infty \). [Hint: Take \( c = g_9g_1g_6, d = g_1g_9g_1g_11 \).]
Exercise 3.28. (action on the Cantor space and the real line)

(i) Let $\hat{A}$ denote the Cantor space consisting of all $\mathbb{N}$-indexed sequences of 0’s and 1’s. For $s \in \hat{A}$ and $a \in G^+_{ld}$, define $a \cdot s$ in $\hat{A}$ as follows: for $s = \beta s_0$, where $\beta$ is the unique prefix of $s$ that belongs to the outline of $t^a_\alpha$, we put $a(s) = \alpha s_0$, where $\alpha$ is the origin of $\beta$ under $a$, i.e., the (unique) address satisfying $\beta \in \text{Heir}\{(\alpha), a\}$. Prove that the mapping $s \mapsto a(s)$ is surjective, and it gives a left action on $\hat{A}$, i.e., $1 \cdot s = s$ and $(ab) \cdot s = a \cdot (b \cdot s)$ hold for every $s$ in $\hat{A}$.

(ii) Prove that this action preserves the order in the sense that $a < b$ holds in $G^+_{ld}$ if and only if there exists $s$ in $\hat{A}$ such that $a \cdot s = b \cdot s$ holds for $s \leq s_0$, but $a \cdot s < b \cdot s$ holds for $s > s_0$, $s$ close enough to $s_0$ (the order on $\hat{A}$ is the lexicographical order).

(iii) Show that carrying the previous action of $G^+_{ld}$ on $\hat{A}$ to the interval $[0, 1)$ of $\mathbb{R}$ using the binary expansion of the reals amounts to associating with every element $a$ of $G^+_{ld}$ a continuous, piecewise affine mapping $f_a$ of $[0, 1)$ into itself. Check that $f_a$ is defined by

$$f_a(x) = \begin{cases} 
2x & \text{for } 0 \leq x < 1/4, \\
 x + 1/4 & \text{for } 1/4 \leq x < 1/2, \\
2x - 1 & \text{for } 1/2 \leq x < 3/4, \\
x & \text{for } 3/4 \leq x < 1.
\end{cases}$$

Prove that $a < b$ holds in $G^+_{ld}$ if and only if there exists a real $x_0$ satisfying $a(x) = b(x)$ for $x \leq x_0$ and $a(x_0 + \varepsilon) < b(x_0 + \varepsilon)$ for $\varepsilon$ small enough.

(iv) Develop a similar study for the group $G_A$ corresponding to associativity. [In this case, the mappings $f_a$ are bijections, and the action is defined on the group, and not only on the monoid.]

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4. Parabolic Subgroups

We have observed in Section 1 that, for every address $\gamma$, the shift mapping $\text{sh}_\gamma$ defines an endomorphism of $G_{ld}$. The aim of this short section is to prove:

**Proposition 4.1. (shift)** For every address $\gamma$, the shift endomorphism $\text{sh}_\gamma$ of $G_{ld}$ is injective.

For $B$ a subset of $A$, let us denote by $G_B$ the subgroup of $G_{ld}$ generated by the elements $g_\alpha$ with $\alpha \in B$—such a subgroup is usually called a parabolic
subgroup in the framework of Artin groups. By construction, the image of \( s_h \gamma \) is the parabolic subgroup of \( G_{LD} \) associated with the subset \( s_h \gamma (A) \). Then Proposition 4.1 (shift) is a counterpart to the result that a parabolic subgroup of an Artin group is the Artin group associated with the induced graph. In the current case, we have proved that the parabolic subgroup associated with the subset \( s_h \gamma (A) \) of \( A \) is isomorphic to \( G_{LD} \), i.e., to the “Artin” group associated with the graph \( A \), which, as a graph, is isomorphic to \( s_h \gamma (A) \).

The method we shall use is the same as for proving the injectivity of the shift mapping on braids (Lemma I.3.3). In the case of braids, the idea was to remove the leftmost strand; in the current case, we shall use a convenient combination of cutting and collapsing to get rid of all addresses that do not begin with \( \gamma \).

**Geometric operators on \( M_{LD} \)**

In Chapter VII, we established that a number of geometric results about the action of the operators \( LD_w \) on terms admit a purely syntactic counterpart. This enables us to define analogs in \( M_{LD} \).

For instance, we proved that, if \( u \) and \( u' \) are words on \( A \) satisfying \( u' \equiv u \), and \( B \) is a subset of \( A \) such that \( \text{Heir}(B, u) \) exists, then \( \text{Heir}(B, u') \) exists as well, and we have \( \text{Heir}(B, u') = \text{Heir}(B, u) \). We can therefore define, for \( a \) in \( M_{LD} \), \( \text{Heir}(B, a) \) to be the common value of \( \text{Heir}(B, u) \) for \( u \) an arbitrary (positive) expression of \( a \), when it exists. Similarly, for \( \alpha \) an address, we define \( \text{heir}(\alpha, a) \) to be the common value of \( \text{heir}(\alpha, u) \) for \( u \) a positive expression of \( a \), when it exists. Then, by construction, we have the induction formulas

\[
\text{Heir}(B, ab) = \text{Heir}(\text{Heir}(B, a), b), \tag{4.1}
\]

\[
\text{heir}(\alpha, ab) = \text{heir}(\text{heir}(\alpha, a), b) \tag{4.2}
\]

when they are defined.

Another example is the cut application of Chapter VII. Applying Proposition VII.4.16 (syntactic cut), we see that the mapping \( u \mapsto \text{cut}(u, \beta) \) induces a well defined mapping on the subset of \( M_{LD} \) made of those elements \( a \) for which \( \text{heir}(\beta, a) \) exists. For \( a \) in \( M_{LD} \), we naturally write \( \text{cut}(a, \beta) \) for the value of \( \text{cut}(u, \beta) \) where \( u \) is an arbitrary positive expression of \( a \). Then we have:

\[
\text{cut}(ab, \beta) = \text{cut}(a, \beta) \text{cut}(b, \text{heir}(\beta, a)), \tag{4.3}
\]

and Proposition VII.4.3 (effective cut) then gives:

**Proposition 4.2. (cut)** Assume that \( a \) belongs to \( M_{LD} \), \( t \) is a term such that \( t \cdot a \) exists, and \( \beta \) is an address in the outline of \( t \). Then we have

\[
\text{cut}(t \cdot a, \text{heir}(\beta, a)) = \text{cut}(t, \beta) \cdot \text{cut}(a, \beta).
\]
Using similarly Proposition VII.4.17 (syntactic collapsing), we see that, for \( a \) in \( M_{LD} \), and \( B \) a set of addresses such that \( \text{Heir}(B, a) \) exists, an element \( \text{coll}(a, B) \) of \( M_{LD} \) can be defined to be the common value of \( \text{coll}(u, B) \) for \( u \) an arbitrary expression of \( a \). Then we have

\[
\text{coll}(ab, B) = \text{coll}(a, B) \text{coll}(b, \text{Heir}(B, a)),
\]

and Proposition VII.4.13 (effective collapsing) gives:

**Proposition 4.3.** (collapsing) Assume that \( a \) belongs to \( M_{LD} \), \( B \) is a set of addresses such that \( \text{Heir}(B, a) \) exists, and \( t \) is a term such that \( t \cdot a \) exists. Then we have

\[
\text{coll}(t \cdot a, \text{Heir}(B, a)) = \text{coll}(t, B) \cdot \text{coll}(a, B).
\]

**Remark 4.4.** Let us define, for \( w \) a (positive) braid word, \( \text{coll}(w, i) \) to be the braid word coding the braid diagram obtained from that of \( w \) by removing the strand initially at position \( i \). Then \( \text{coll}(\cdot, i) \) induces a well defined mapping on \( B_\infty \), and we have for all \( a, b \) in \( B_\infty \)

\[
\text{coll}(ab, i) = \text{coll}(a, i) \text{coll}(b, \text{perm}(a)^{-1}(i)),
\]

a formula which is directly comparable with (4.4) above. Using (4.5), it is easy to give a purely algebraic proof of the injectivity of \( \text{sh} \) on \( B_\infty \), similar to the proof of the injectivity of \( \text{sh}_\gamma \) on \( G_{LD} \) that will be given below. Formulas (4.3) and (4.4)—as well as (4.5) in the case of braids—show that the mappings \( \text{cut} \) and \( \text{coll} \) are sort of skew endomorphisms on \( M_{LD} \).

**Injectivity of shift**

We are now ready to establish Proposition 4.1 (shift).

**Proof.** Assume that \( w, w' \) are words on \( A \cup A^{-1} \), and \( \gamma w \equiv \gamma w' \) holds. By Proposition 1.14 (fraction), there exist positive words \( v, v' \) satisfying

\[
N(\gamma w) \cdot v \equiv^+ N(\gamma w') \cdot v' \quad \text{and} \quad D(\gamma w) \cdot v \equiv^+ D(\gamma w') \cdot v'.
\]

By construction, the latter relations are also

\[
\gamma N(w) \cdot v \equiv^+ \gamma N(w') \cdot v' \quad \text{and} \quad \gamma D(w) \cdot v \equiv^+ \gamma D(w') \cdot v'. \tag{4.6}
\]

Our aim is to eliminate the initial \( \gamma \)'s in (4.6). To this end, we use the mappings \( \text{cut} \) and \( \text{coll} \). First, there exists an integer \( p \) such that both

\[
\text{heir}(\gamma P, \gamma N(w) \cdot v) \quad \text{and} \quad \text{heir}(\gamma P^2, \gamma D(w) \cdot v)
\]
exist: indeed, the only possible obstruction for such addresses to exist is $\gamma 1^p$ being too short. Let us then apply the mapping $\text{cut}(:, \gamma 1^p)$ to (4.6). By definition, we have

$$\text{cut}(\gamma N(w) \cdot v, \gamma 1^p) = \text{cut}(\gamma N(w), \gamma 1^p) \cdot \text{cut}(v, \text{heir}(\gamma 1^p, \gamma N(w))).$$

Now, by construction, we have $\text{cut}(\gamma N(w), \gamma 1^p) = 1^m(N(w))$, where $m$ is the number of 1’s in $\gamma$, and $\text{heir}(\gamma 1^p, \gamma N(w)) = 1^m$. Thus we deduce

$$\gamma_1 N(w) \cdot v_1 \equiv \gamma_1 N(w') \cdot v'_1, \quad \text{and} \quad \gamma_1 D(w) \cdot v_1 \equiv \gamma_1 D(w') \cdot v'_1, \quad (4.7)$$

with $\gamma_1 = 1^m$, $v_1 = \text{cut}(v, 1^m)$ and $v'_1 = \text{cut}(v', 1^m)$. So, at this point, we have eliminated all 0’s in the address $\gamma$.

We now resort to collapsing. As above, we can find an integer $q$ large enough to make sure that, letting $B = \{01^q, 101^q, 1101^q, \ldots, 1^{m-1}01^q\}$, the sets

$$\text{Heir}(B, 1^m N(w) \cdot v_1) \quad \text{and} \quad \text{Heir}(B, 1^m D(w) \cdot v_1)$$

exist. Then, we have

$$\text{coll}(\gamma_1 N(w) \cdot v_1, B) = \text{coll}(\gamma_1 N(w), B) \cdot \text{coll}(v_1, \text{Heir}(B, \gamma_1 N(w))).$$

We have $\text{coll}(\gamma_1 N(w), B) = N(w)$, and $\text{Heir}(B, \gamma_1 N(w)) = B$ by definition. We deduce from (4.7)

$$N(w) \cdot v_2 \equiv N(w') \cdot v'_2 \quad \text{and} \quad D(w) \cdot v_2 \equiv D(w') \cdot v'_2 \quad (4.8)$$

with $v_2 = \text{coll}(v_1, B)$ and $v'_2 = \text{coll}(v'_1, B)$. Now, (4.8) implies $w \equiv w'$.

**Exercise 4.5. (kernel)** (i) Prove that the kernel of the projection of $G_{LD}$ onto $B_{\infty}^*$ includes the direct product of infinitely many copies of $G_{LD}$. [Hint: $\text{Ker}(\text{pr})$ includes $\prod_i \text{sh}_{1i0}(G_{LD})$; apply Proposition 4.1.]

(ii) Define $\text{pr}^{-1}(\sigma_i) = 1^{i-1}$, and extend $\text{pr}^{-1}$ into an alphabetical homomorphism of $BW_{\infty}^+$ into $A^*$. Prove that, for every $u$ in $A^*$, we have $u \equiv \text{pr}^{-1}(\text{pr}(u)) \cdot \prod_i 1^0 u_i$ for some $u_1, u_2, \ldots$ in $A^*$.

(iii) Assume $u, v \in BW_{\infty}^+$ with $u \equiv v$. Show that $\text{pr}^{-1}(u) \cdot \prod_i 1^0 u_i \equiv \text{pr}^{-1}(v) \cdot \prod_i 1^0 v_i$ holds for some $u_1, u_2, v_1, v_2, \ldots$ in $A^*$.

(iv) Assume $\text{pr}(w) \equiv \varepsilon$, where $w$ is a word on $A \cup A^{-1}$. Show that $w \equiv \varepsilon \cdot \prod_i 1^0 w_i \cdot u^{-1}$ holds for some words $u, w_1, w_2, \ldots$ with $u$ positive. [Hint: Apply (ii) and (iii) to $N(w)$ and $D(w)$.] Deduce that every element of $G_{LD}$ in $\text{Ker}(\text{pr})$ can be expressed as $aba^{-1}$ where $a$ belongs to the submonoid generated by the elements $g_1$, and $b$ belongs to $\prod_i \text{sh}_{1i0}(G_{LD})$. 
Exercise 4.6. (left subterm) For $a$ in $M_{LD}$ and $p \geq 0$, define $\text{dil}(a, p)$ in $\mathbb{N}$ and $\text{left}(a, p)$ in $M_{LD}$ to be the common values of $\text{dil}(u, p)$ and $\text{left}(u, p)$ respectively for $u$ representing $a$. Prove the equality $\text{left}(ab, p) = \text{left}(a, p) \text{left}(b, \text{dil}(p, a))$ Prove that, for $a \in M_{LD}$, and $t$ a term such that $t \cdot a$ and $\text{left}^p(t)$ exist, we have $\text{left}^\text{dil}(p, a)(t \cdot a) = \text{left}^p(t) \cdot \text{left}(a, p)$.

Exercise 4.7. (alternative proof) Prove that $\text{LD}_w$ and $\text{LD}_{w'}$ agreeing on one term implies $w \equiv w'$, using Proposition 3.23 (action I) instead of Proposition 2.14 (action II). [Use the injectivity of $\text{sh}_0$.]

Exercise 4.8. (decomposition) Assume that $B$ is a finite set of pairwise orthogonal addresses. Assume $\prod_{\alpha \in B} \alpha u_\alpha \equiv^+ \prod_{\alpha \in B} \alpha v_\alpha$, where all $u_\alpha$, $v_\alpha$ are words on $A$. Prove $u_\alpha \equiv^+ v_\alpha$ for each $\alpha$ in $B$.

5. Simple Elements in $M_{LD}$

Simple braids play a significant role in the study of the monoid $B_+^\infty$ in Chapter II. In this section, we develop the notion of a simple element in the monoid $M_{LD}$. The intuition is similar in both cases, and so are the results. Actually, the simple elements in $M_{LD}$ are mapped to simple braids under the projection of $M_{LD}$ onto $B_+^\infty$, and all results we shall establish in $M_{LD}$ re-prove their braid counterpart. The main distinction is that $B_+^\infty$ is the union of the finitely generated monoids $B_n^+$, and, in each of them, there exists a Garside element, namely $\Delta_n$. This is not true in $M_{LD}$, but we shall see that the class $\Delta_t$ of the word $\Delta_t$ can be used as a counterpart to the braid $\Delta_n$.

In the case of $B_n^+$, we have introduced a special family of positive braid words, called permutation braid words, and the main result is that a braid $b$ can be represented by a permutation braid word if and only if it is a divisor of $\Delta_n$, and if and only if it can be represented by a braid diagram where any two strands cross at most once. Our approach in the case of $M_{LD}$ will be similar: we shall introduce the notion of a permutation-like word on $A$ by an explicit definition, and the notion of a simple element of $M_{LD}$ by a geometrical property of the corresponding LD-operators. Then we prove the following result, which is directly reminiscent of Proposition II.4.14 (simple)—which it implies:

Proposition 5.1. (simple) For $a$ in $M_{LD}$, the following are equivalent:

(i) The element $a$ is permutation-like;
(ii) The element $a$ is simple;
(iii) There exists a term $t$ such that $a$ is a left divisor of $\Delta_t$.
Moreover, these conditions are consequences of
(iv) There exists a term $t$ such that $a$ is a right divisor of $\Delta_t$.

As an application, we shall deduce a unique normal form in $M_{LD}$.

**Permutation-like elements**

Let us recall from Chapter VII that, for $\alpha$ in $A$ and $p \geq 0$, $\alpha^{(p)}$ is defined to be the word $\alpha 1^{p-1} \cdot \alpha 1^{p-2} \cdot \cdots \cdot \alpha 1 \cdot \alpha$ for $p \geq 1$, and to be $\varepsilon$ for $p = 0$. We also recall that, for $\alpha, \beta$ in $A$, $\alpha > \beta$ means that $\alpha$ is a proper prefix of $\beta$, or $\alpha$ lies on the right of $\beta$: thus, for instance, $\phi > 1 > 0$ holds.

**Definition 5.2. (permutation-like, exponent)** We say that a word $u$ on $A$ is a permutation-like word if $u$ can be factorized as $\alpha_1^{(p_1)} \cdot \cdots \cdot \alpha_\ell^{(p_\ell)}$ with $\alpha_1 > \cdots > \alpha_\ell$; then, for $\alpha \in A$, the exponent $e(\alpha, u)$ of $\alpha$ in $u$ is defined to be the integer $p$ such that $\alpha^{(p)}$ appears in $u$, if it exists, and to be 0 otherwise. An element of $M_{LD}$ is said to be a permutation-like element if it can be represented by a permutation-like word.

As $\alpha^{(0)}$ has been defined to be the empty word, a permutation-like word can be written as $\prod_{\alpha \in A}^{\alpha^{(p_\alpha)}}$, where $(p_\alpha : \alpha \in A)$ is a sequence of nonnegative integers with finitely many positive entries only. A length 1 word, i.e., a single address, is a permutation-like word. By definition, the projection of a permutation-like element of $M_{LD}$ on $B_{\infty}^+$ is a permutation braid.

**Example 5.3. (permutation-like)** Let $u = 11 \cdot 1 \cdot \phi \cdot 1 \cdot 001 \cdot 00$. We have $u = (11 \cdot 1 \cdot \phi) \cdot (1) \cdot (001 \cdot 00) = \phi^{(3)} \cdot 1^{(1)} \cdot 00^{(2)}$, and $\phi > 1 > 0$. So $u$ is a permutation-like word, with $e(\phi, u) = 3$, $e(0, u) = 0$, and $e(1, u) = 1$.

By definition, every permutation-like word has the form $\phi^{(p)} \cdot 1u_2 \cdot 0u_1$, where $u_1$ and $u_0$ are permutation-like words: this will enable us to develop inductive arguments.

**Lemma 5.4.** A permutation-like element in $M_{LD}$ admits a unique representation by a permutation-like word: if $a$ is a permutation-like element, the unique permutation-like word representing $a$ depends on the operator $LD_a$ only.
Proof. Assume that $u$ is a permutation-like word. We show that the exponents $e(\alpha, u)$ are determined by the operator $LD_u$ using induction on the size of $t^*_u$. If $t^*_u$ is a variable, we have $LD_u = id$, so $u = \varepsilon$, and the result is true. Otherwise, assume $u = \phi^{(p)} \cdot t_1 u_2 \cdot 0 u_1$. Let $t = t^*_u$. Since $t^*_u$ belongs to the domain of $LD_{(p)}$, its right height is at least $p + 1$. Let $t_1 t_2 = t^*_u \cdot \phi^{(p)}$, and $x_f(1) = var(sub(t^*_u, 1^0))$. By construction, we have $var(t_1) = x_f(1)$. Now we have $t^*_u = t^*_u \cdot u = (t_1 \cdot u_1) \cdot (t_2 \cdot u_2)$, we deduce $x_f(p) = var(sub(t^*_u, 0))$. This shows that $t^*_u$ determines $p$, and, therefore, so does $u$. Then, for $e = 1, 2$, the term $t_e$ belongs to the domain of $LD_{u_e}$; it is injective, and it is smaller than $t^*_u$. As $t_e$ is a substitute of $t^*_u$, the latter term is smaller than $t^*_u$ as well. Hence, by induction hypothesis, $t_e \cdot u_e$ determines the exponents in $u_e$, and so does $t \cdot u$, as $t_e \cdot u_e = sub(t \cdot u, e)$ holds. 

It follows that, for every permutation-like element $a$ and every address $\alpha$, we can define without ambiguity the exponent of $\alpha$ in $a$ as the exponent of $\alpha$ in the unique permutation-like word that represents $a$.

### Simple elements

We recall from Chapter VII that, if $t$ is a term, the variable $x_i$ is said to cover the variable $x_j$ in $t$ if there exist two addresses $\alpha$, $\beta$ in the outline of $t$ such that $x_i$ occurs at $\alpha$ in $t$, $x_j$ occurs at $\beta$ in $t$, and $\alpha$ covers $\beta$.

**Definition 5.5.** (semi-injective) The term $t$ is said to be semi-injective if no variable covers itself in $t$.

For a term $t$ to be semi-injective means that, for every subterm $s$ of $t$, the rightmost variable of $s$ occurs only once in $s$. Thus every injective term is semi-injective, but the converse is not true: for instance, the term $(x_1 \cdot x_2) \cdot (x_1 \cdot x_3)$, which is not injective since $x_1$ occurs twice, is semi-injective.

Non-semi-injective terms have good closure properties.

**Lemma 5.6.** Non semi-injective terms are closed under substitution and LD-expansion.

*Proof.* Assume that $t$ is non-semi-injective. Then some variable $x_i$ occurs both at $\alpha 1^r$ and $\alpha 0 \beta$ in $t$. Let $h$ be an arbitrary substitution, and let $x_k = var(h(x_i)), q = ht_0(h(x_i))$. Then $x_k$ occurs at $\alpha 1^{r+q}$ and $\alpha 0 31^q$ in $t^k$. Hence $t^k$ is not semi-injective. On the other hand, we have seen in Lemma VII.1.28 that LD-expansions never delete coverings: if $x_i$ covers $x_j$ in $t$, it covers $x_j$ in every LD-expansion of $t$. This applies in particular when $x_i$ covers itself. 

We introduce a semantic notion of simplicity that is analogous to the condition that any two strands cross at most once in a braid diagram.
Definition 5.7. (simple) For $a$ in $M_{LD}$, we say that $a$ is simple if the image of $LD_a$ contains at least one semi-injective term. For $u$ in $A^*$, we say that $u$ is simple if the class of $u$ in $M_{LD}$ is simple.

Lemma 5.8. Assume that $a$ is an element of $M_{LD}$. Then, the following are equivalent:

(i) The element $a$ is simple;
(ii) The term $t^a_R$ is semi-injective;
(iii) The operator $LD_a$ maps every injective term to a semi-injective term.

Proof. The term $t^a_L$ is injective, and the operator $LD_a$ maps $t^a_L$ to $t^a_R$, so (iii) implies (ii), and (ii) implies (i). Assume (i). Let $t$ be a term in the domain of $LD_a$ such that $t \cdot a$ exists and is semi-injective. By Proposition VII.1.10 (domain), there exists a substitution $h$ satisfying $t = (t^a_L)^h$ and $t \cdot a = (t^a_R)^h$. By Lemma 5.6, $(t^a_R)^h$ being semi-injective implies $t^a_R$ being semi-injective, so (ii) holds. Assume now (ii), and let $t$ be an injective term in the domain of $LD_a$. Then there exists a substitution $h$ satisfying $t = (t^a_R)^h$, and $t$ being injective means that we can assume that $h$ is an injective substitution. Now we have $t \cdot a = (t^a_R)^h$, and such a term being not semi-injective would imply $t^a_R$ itself being not semi-injective.

Using the closure properties of non-semi-injective terms, we obtain the following closure property for simple elements of $M_{LD}$. Observe that the corresponding result for permutation-like elements is not clear—a situation parallel to what happened in the case of simple braids and permutation braids.

Lemma 5.9. Every divisor of a simple element of $M_{LD}$ is simple.

Proof. Assume that $a$ is not simple, and let $b, c$ be arbitrary elements of $M_{LD}$. By Proposition VII.1.10 (domain), the term $t^a_{ba}$ is a substitute of $t^a_R$, and the term $t^a_{bac}$ is an LD-expansion of the previous term. By hypothesis, $t^a_R$ is not semi-injective, hence, by Lemma 5.6, $t^a_{ba}$ and $t^a_{bac}$ are not semi-injective either. Hence $bac$ is not simple.

We shall prove eventually that permutation-like elements and simple elements in $M_{LD}$ coincide. For the moment, let us observe that one direction is easy.

Lemma 5.10. Every permutation-like element of $M_{LD}$ is simple.

Proof. We show that using induction on the size of the term $t^u_L$ that, if $u$ is a permutation-like word, then $u$ is simple. By definition, we have $u = \phi^{(p)} \cdot 1u_2 \cdot 0u_1$ where $u_1$ and $u_2$ are permutation-like words. Let $t = t^u_L$. Then
Section VIII.5: Simple Elements in $M_{LD}$

$t \cdot \phi^{(p)}$ exists, and, therefore, the right height of $t$ is at least $p + 1$, i.e., $t$ can be written as $t_1 \cdots t_{p+2}$. Then we have $t \cdot \phi^{(p)} = t'_1 \cdot t'_2$, with

$$t'_1 = t_1 \cdots t_p \cdot t_{p+1} \quad \text{and} \quad t'_2 = t_1 \cdots t_p \cdot t_{p+2}.$$  

By hypothesis, for $c = 1, 2$, $t'_c$ is an injective term that lies in the domain of the operator $LD_{uc}$. The terms $t'_1$ and $t'_2$ are smaller than $t$, so, $t'u_1$ and $t'u_2$ are smaller than $t'_c$. By induction hypothesis, the LD-expansions $t'_1 \cdot u_1$ and $t'_2 \cdot u_2$ are semi-injective terms. Hence $t \cdot u$, which is $(t'_1 \cdot u_1) \cdot (t'_2 \cdot u_2)$, is semi-injective as well, for the rightmost variable of $t'_2 \cdot u_2$, which is $\text{var}_n(t'_2)$, occurs neither in $t'_1$ nor in $t'_1 \cdot u_1$.

Example 5.11. (not permutation-like) We have obtained a criterion for proving that an element of $M_{LD}$ is not permutation-like. For instance, $g^{-} \cdot g^{-}$ is not simple, and, therefore, it is not permutation-like: indeed we have $(x_1 \cdot x_2 \cdot x_3) \cdot \phi^2 = ((x_1 \cdot x_2) \cdot x_1) \cdot ((x_1 \cdot x_2) \cdot x_3)$, a non-semi-injective term since the variable $x_1$ occurs both at 01 and 000.

Product of permutation-like elements

Our goal is now to establish the converse of Lemma 5.10. We begin with a series of computation formulas. The point is to determine the permutation-like decomposition of the products $\alpha^{(p)} \cdot \phi^{(q)}$, when it exists. Two cases arise, according to whether $\alpha$ contains at least one 0 or not.

Lemma 5.12. Assume $\alpha = 1^m0\beta$. Then $\alpha^{(p)} \cdot \phi^{(q)}$ is simple for all $p, q$, and we have

$$\alpha^{(p)} \cdot \phi^{(q)} \equiv^+ \begin{cases} \phi^{(q)} \cdot \alpha^{(p)} & \text{for } q < m, \\ \phi^{(q)} \cdot (01^m\beta)^{(p)} & \text{for } q = m, \\ \phi^{(q)} \cdot (1\alpha)^{(p)} & \text{for } q > m. \end{cases}$$

Proof. Assume $p = 1$. For $m \geq q + 1$, $1^m0\beta$ commutes with every factor in $\phi^{(q)}$ by type 11 relations, so $\alpha$ commutes with $\phi^{(q)}$. For $m = q$, using $q$ successive type 10 relations, we obtain

$$\alpha \cdot \phi^{(m)} = 1^m0\beta \cdot 1^{m-1} \cdots \cdot \phi \equiv^+ 1^{m-1} \cdot 1^{m-1}01\beta \cdot 1^{m-2} \cdots \cdot \phi \cdots \equiv^+ 1^{m-1} \cdots 1 \cdot 101^{m-1}\beta \cdot \phi \equiv^+ 1^{m-1} \cdots 1 \cdot \phi \cdot 01^{m}\beta = \phi^{(m)} \cdot 01^{m}\beta.$$
For \( m < q \), we find

\[
\alpha \cdot \phi^{(q)} = 1^m 0 \beta \cdot (1^{m+1}(q-m-1) \cdot 1^m \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^{m+1}(q-m-1) \cdot 1^m 0 \beta \cdot 1^m \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^{m+1}(q-m-1) \cdot 1^m \cdot 1^m 10 \beta \cdot 1^m 00 \beta \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^{m+1}(q-m-1) \cdot 1^m \cdot 1^m 10 \beta \cdot \phi^{(m)} \cdot 01^m 0 \beta
\]

\[
\equiv^+ (1^{m+1}(q-m-1) \cdot 1^m \cdot \phi^{(m)} \cdot 1^m 10 \beta \cdot 01^m 0 \beta = \phi^{(q)} \cdot 1 \alpha \cdot 0 \alpha.
\] (11)

Extending the result to the case \( p > 1 \) is easy in the first two cases. In the last case, we observe that \( \alpha 1^p - 1 \cdot 0 \alpha 1^{p-1} \cdot \ldots \cdot 1 \alpha \cdot 0 \alpha \) is equivalent to \((1\alpha)^{(p)} \cdot (0\alpha)^{(p)}\) using type \( \perp \) relations.

**Lemma 5.13.** Assume \( \alpha = 1^m \). Then \( \alpha^{(p)} \cdot \phi^{(q)} \) is simple unless \( m < q \leq m + p \) holds, and we have

\[
\alpha^{(p)} \cdot \phi^{(q)} \equiv^+ \begin{cases} 
\phi^{(q)} \cdot \alpha^{(p)} & \text{for } q < m, \\
\phi^{(p+q)} & \text{for } q = m, \\
\phi^{(q)} \cdot (1^{m+1})^{(p)} \cdot (01^m)^{(p)} & \text{for } q > m + p.
\end{cases}
\]

**Proof.** For \( q < m \), every factor in \( \phi^{(q)} \) commutes with every factor in \( \alpha^{(p)} \) by type 11 relations, so \( \alpha^{(p)} \) and \( \phi^{(q)} \) commute. For \( q = m \), we have \( \alpha^{(p)} \cdot \phi^{(q)} = \phi^{(p+q)} \) by definition. For \( q > m + p \), we use induction on \( p \). For \( p = 1 \), hence \( q \geq m + 2 \), we find

\[
1^m \cdot \phi^{(q)} = 1^m \cdot (1^{m+2}(q-m-2) \cdot 1^{m+1} \cdot 1^m \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^{m+2}(q-m-2) \cdot 1^m \cdot 1^{m+1} \cdot 1^m \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^{m+2}(q-m-2) \cdot 1^m \cdot 1^{m+1} \cdot 1^m \cdot 1^m \cdot \phi^{(m)}
\]

\[
= (1^m)^{(q-m)} \cdot 1^{m+1} \cdot 1^m \cdot \phi^{(m)}
\]

\[
\equiv^+ (1^m)^{(q-m)} \cdot 1^{m+1} \cdot \phi^{(m)} \cdot 01^m
\]

\[
\equiv^+ (1^m)^{(q-m)} \cdot \phi^{(m)} \cdot 1^{m+1} \cdot 01^m = \phi^{(q)} \cdot 1^{m+1} \cdot 01^m.
\] (Lemma 5.12)

Assume now \( p > 1 \). We have

\[
(1^m)^{(p)} \cdot \phi^{(q)} = (1^{m+1}(p-1) \cdot 1^m \cdot \phi^{(q)}
\]

\[
\equiv^+ (1^{m+1}(p-1) \cdot \phi^{(q)} \cdot 1^{m+1} \cdot 01^m
\]

\[
\equiv^+ \phi^{(q)} \cdot (1^{m+2}(p-1) \cdot (01^m)^{(p-1)} \cdot 1^m \cdot 01^m
\]

\[
\equiv^+ \phi^{(q)} \cdot (1^{m+2}(p-1) \cdot 1^{m+1} \cdot (01^m)^{(p-1)} \cdot 01^m
\]

\[
= \phi^{(q)} \cdot (1^{m+1})^{(p)} \cdot (01^m)^{(p)}.
\]

The previous explicit formulas show that, in the three previous cases, \( \alpha^{(p)} \cdot \phi^{(q)} \) is a permutation-like element. So it only remains to prove that the product is not simple in the case \( m < q \leq m + p \). Owing to Lemma 5.10, it suffices to exhibit an injective term whose image under the operator \( LD_{\alpha^{(p)}} \cdot LD_{\phi^{(q)}} \) is not
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semi-injective. Let $t = x_1 \cdots x_{m+p+2}$. Then $t \cdot \alpha^{(p)}$ exists, and it is

$$x_1 \cdots x_m \cdot (x_{m+1} \cdots x_{m+p+1}) \cdot x_{m+1} \cdots x_{m+p+2}.$$  

Applying $LD_{\phi(q)}$ to this term gives a term whose $01^m$-subterm is

$$(x_{m+1} \cdots x_{m+p+1}) \cdot x_{m+1} \cdots x_{q+1},$$

and the rightmost variable of this subterm, namely $x_{q+1}$, also occurs in its left subterm, so it is not semi-injective—see an example in Figure 5.1.

\[ \begin{array}{c}
\text{LD}_{(2)} \rightarrow \\
\text{LD}_{(3)} \rightarrow
\end{array} \]

\[ \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_{4} \ldots
\end{array} \quad \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_{3x_4} \\
x_{3} \ldots
\end{array} \quad \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_{2x_4} \\
x_{3x_4} \\
x_{2} \ldots
\end{array} \]

Figure 5.1: A non-simple case: $m = 1$, $p = 2$, $q = 3$.

We can now determine whether a permutation-like element remains a permutation-like element when an additional factor $\phi(q)$ is added.

**Lemma 5.14.** Assume that $u$ is a permutation-like word, and $q \geq 0$ holds. Let $r = q + e(1^q, u)$. Then $u \cdot \phi(q)$ is simple if and only if $m + e(1^m, u) < r$ holds for $0 \leq m < q$; in this case, $u \cdot \phi(q)$ is a permutation-like element, and we have $e(\phi, u \cdot \phi(q)) = r$.

**Proof.** Decompose $u$ as

$$\prod_{m=0}^{\infty} (1^m)^{(p_m)} \cdot \prod_{m=\infty}^{0} 1^m 0 u_m,$$

where all $u_m$ are permutation-like words. We add the factor $\phi(q)$ on the right, and try to push this factor to the left and integrate it in the decomposition. By Lemma 5.12, we cross the right product: $u \cdot \phi(q)$ is $\equiv \perp$-equivalent to

$$\prod_{m=0}^{\infty} (1^m)^{(p_m)} \cdot \phi(q) \cdot \prod_{m=\infty}^{q+1} 1^m 0 u_m \cdot 0^1 q u_q \cdot \prod_{m=q-1}^{0} (1^{m+1} 0 u_m \cdot 0^1 q u_q),$$

hence, using type $\perp$ relations, to

$$\prod_{m=0}^{\infty} (1^m)^{(p_m)} \cdot \phi(q) \cdot \prod_{m=\infty}^{q+1} 1^m 0 u_m \cdot \prod_{m=q-1}^{0} 1^{m+1} 0 u_m \cdot 0^1 q u_q \cdot \prod_{m=q-1}^{0} 1^{m} 0 u_m.$$
It remains to study the expression \( \prod_{m=0}^{\infty} (1^m)_{(p_m)} \cdot \phi^{(q)} \). We use now Lemma 5.13 to push \( \phi^{(q)} \) to the left. First, we have

\[
\prod_{m=q}^{\infty} (1^m)_{(p_m)} \cdot \phi^{(r)} \equiv^+ \prod_{m=q+1}^{\infty} (1^m)_{(p_m)},
\]

with \( r = q + p_q \), i.e., \( r = q + e(1^q, a) \), and we are left with \( \prod_{m=0}^{q-1} (1^m)_{(p_m)} \cdot \phi^{(r)} \).

By Lemma 5.13, two cases are possible. Either the condition \( q - 1 + p_q - 1 \geq r \) holds, and then \((1^{q-1})_{(p_q-1)} \cdot \phi^{(r)} \) is not simple, and, therefore, by Lemma 5.9, \( u \cdot \phi^{(q)} \) is not simple either. Or \( q - 1 + p_q - 1 < r \) holds, and \((1^{q-1})_{(p_q-1)} \cdot \phi^{(r)} \) is \( \equiv^+ \)-equivalent to \( \phi^{(r)} \cdot (1^q)_{(p_q-1)} \cdot (01^{q-1})_{(p_q-1)} \). We continue with the product \((1^{q-1})_{(p_q-1)} \cdot \phi^{(r)} \). Again two cases are possible: in the one case, \( u \cdot \phi^{(q)} \) is not simple, in the other, it is, we can push the factor \( \phi^{(q)} \) to the left, and the process continues. Finally, if the condition \( m + p_m < r \) fails for some \( m \), \( u \cdot \phi^{(q)} \) is not simple; if the condition holds for every \( m \), the factor \( \phi^{(q)} \) migrates to the leftmost position, and we obtain that \( u \cdot \phi^{(q)} \) is \( \equiv^+ \)-equivalent to

\[
\phi^{(r)} \cdot \prod_{m=0}^{q-1} (1^{m+1})_{(p_m)} \cdot \prod_{m=0}^{q-1} (01^m)_{(p_m)} \cdot \prod_{m=\infty}^{q+1} 1^m 0 u_m \cdot \prod_{m=q}^{0} 1^{m+1} 0 u_m, \cdot \prod_{m=q-1}^{0} 01^m 0 u_m,
\]

which can be rearranged using relations of type \( \perp \) and renumbering into

\[
\phi^{(r)} \cdot \prod_{m=1}^{q} (1^m)_{(p_m-1)} \cdot \prod_{m=\infty}^{q+1} 1^m 0 u_m \cdot \prod_{m=0}^{1} 1^m 0 u_{m-1} \cdot \prod_{m=0}^{q-1} 1^m 0 u_m \cdot \prod_{m=q}^{0} 01^m 0 u_m.
\]

an explicit permutation-like element of \( M_{\mathcal{L}D} \).

**Proposition 5.15.** (equivalence) An element of \( M_{\mathcal{L}D} \) is a permutation-like element if and only if it is simple.

**Proof.** We have already seen that every permutation-like element is simple. We establish now, using induction on the size of \( t_u^p \), that \( u \) being a simple word implies \( u \) being \( \equiv^+ \)-equivalent to a permutation-like word. If \( t_u^p \) is a variable, we have \( u = \varepsilon \), a permutation-like word. Otherwise, \( u \) can be decomposed as \( v \cdot \alpha^{(q)} \) with \( q \geq 1 \). By Lemma 5.9, \( v \) is simple, so, by induction hypothesis, it is \( \equiv^+ \)-equivalent to some permutation-like word \( v' \). We show inductively on the length of \( \alpha \) that \( v' \cdot \alpha^{(q)} \) is \( \equiv^+ \)-equivalent to some permutation-like word. For \( \alpha = \phi \), the previous lemma gives the result. Otherwise, assume \( \alpha = e(\beta) \),
Section VIII.5: Simple Elements in $M_{LD}$

with $e = 0$ or $e = 1$. There exist an integer $p$ and permutation-like words $v_1, v_0$ satisfying $v' \equiv \alpha(q)$ is simple, which implies that $v_e \cdot \beta(q)$ is simple too, since a subterm of a semi-injective term is semi-injective. By induction hypothesis, $v_e \cdot \beta(q)$ is $\equiv$-equivalent to some permutation-like word. Hence, so are $ev_e \cdot \alpha(q)$, and $ev_e \cdot \alpha(q)$, hence $u$. ■

Simple LD-expansions

In the study of LD-expansions, and, in particular, in the proof of confluence of Chapter V, the key step is to introduce for each term $t$ the distinguished LD-expansion $\partial t$ of $t$, which is a common LD-expansion of all basic LD-expansions of $t$. Here we introduce the natural notion of a simple LD-expansion, and we show that $\partial t$ is a maximal simple LD-expansion of $t$, this corresponding to the element of $M_{LD}$ represented by $\Delta_t$, being a maximal simple element in $M_{LD}$—as $\Delta_n$ is a maximal simple braid in $B_n^+$. 

Definition 5.16. (element $\Delta_t$) For $t$ a term, the class of the word $\Delta_t$ in $M_{LD}$ is denoted $\Delta_t$. Similarly, the class of $\Delta_k(t)$ is denoted $\Delta_k(t)$.

Definition 5.17. (simple expansion) We say that $t'$ is a simple LD-expansion of $t$ if $t' = t \cdot u$ holds for some simple word $u$.

By Lemma 5.4, there exists a one-to-one correspondence between the simple LD-expansions of a term $t$ and the permutation-like words $u$ such that $t$ belongs to the domain of $LD_u$.

Proposition 5.18. (maximal simple) For every term $t$, the term $\partial t$ is the maximal simple LD-expansion of $t$, and $\Delta_t$ is the (unique) permutation-like word $u$ such that $LD_u$ maps $t$ to $\partial t$.

Proof. We already know that $LD_{\Delta_t}$ maps $t$ to $\partial t$. That $\Delta_t$ is a permutation-like word follows from its explicit definition. We prove using induction on $t$ that no LD-expansion of $\partial t$ may be a semi-injective term. The result is vacuously true for $t$ a variable. Assume $t = t_1 \cdot t_2$. We consider first LD-expansion at $\phi$. The equality $\partial t = \partial t_1 \cdot \partial t_2$ shows that every variable occurring in $t$ except possibly the rightmost one occurs both in the left and the right subterm of $\partial t$. So the rightmost variable of sub($\partial t$, 10), say $x_i$, occurs in sub($\partial t$, 0) also, hence, when $LD_{\phi}$ is applied to $\partial t$, $x_i$ covers itself in the resulting LD-expansion, which therefore is not semi-injective. Consider now LD-expansion at $\alpha$, where $\alpha$ is a nonempty address, say $\alpha = e\beta$ with $e = 0$ or $e = 1$. We have $\partial t = \partial s_1 \cdot \partial s_2$, with $s_1 \cdot s_2 = t \cdot \phi^{(r)}$ for some $r$. By construction, we have sub($\partial t \cdot \alpha$, $e$) = $\partial s_e \cdot \beta$, and, $s_e$ is smaller than $t$, the induction hypothesis implies that $\partial s_e \cdot \beta$ is not semi-injective. So $t \cdot \alpha$ is not semi-injective either. ■
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Corollary 5.19. For every term \( t \), the element \( \Delta_t \) is simple, and it is maximal in the sense that \( \Delta_t a \) is simple for no element \( a \) such that \( t \cdot \Delta_t a \) exists.

Proposition 5.20. (divisors of \( \Delta \)) For every \( a \) of \( M_{LD} \), the following are equivalent:

(i) The element \( a \) is simple;
(ii) There exists a term \( t \) such that \( a \) is a left divisor of \( \Delta_t \) in \( M_{LD} \);
(iii) For every term \( t \) such that \( t \cdot a \) exists, \( a \) is a left divisor of \( \Delta_t \) in \( M_{LD} \).

Proof. By definition, (iii) implies (ii), and, by Lemma 5.9, (ii) implies (i) as \( \Delta_t \) is simple. So the point is to prove that (i) implies (iii). We prove using induction on \( t \) that, if \( a \) is a permutation-like element of \( M_{LD} \) and \( t \cdot a \) is defined, then \( a \) is a left divisor of \( \Delta_t \). The result is true when \( t \) is a variable. Otherwise, let \( r + 1 \) be the right height of \( t \). By definition, \( t \) belongs to the domain of \( LD_a \), so the inequality \( m + e(1^m, a) \leq r \) holds for every \( m \) between 0 and \( r \), and there exists a least \( q \) satisfying \( q + e(1^q, a) = r \). By Lemma 5.14, we deduce that \( a \cdot \phi(q) \) is simple, with \( e(\phi, a \cdot \phi(q)) = r \). This means that there exist simple elements \( a_1, a_2 \) satisfying

\[
 a \cdot \lambda(q) = \lambda(r) \cdot 1a_2 \cdot 0a_1.
\]

By construction, the term \( t \) belongs to the domain of \( LD_{\phi(q)} \). Let \( s_1, s_2 \) be its image. By definition, the term \( s_e \cdot a_e \) is defined for \( e = 1, 2 \), and, by construction, \( s_e \) is smaller than \( t \). Hence, by induction hypothesis, there exists an element \( b_e \) of \( M_{LD} \) satisfying \( a_e b_e = \Delta_{s_e} \). We deduce

\[
 a \cdot \phi(q) \cdot 1b_2 \cdot 0b_1 = \phi(r) \cdot 1a_2 \cdot 0a_1 \cdot 1b_2 \cdot 0b_1 = \phi(r) \cdot 1\Delta_{s_2} \cdot 0\Delta_{s_1} = \Delta_t.
\]

We can now complete the proof of Proposition 5.1 (simple).

Proof. First, (i) implying (ii) is Lemma 5.10; (ii) implying (i) is Proposition 5.15 (equivalence). Then \( \Delta_t \) is simple, so, by Lemma 5.9, (iii) and (iv) imply (ii). Finally (ii) implies (iii) by Proposition 5.20 (divisors of \( \Delta \)).

We do not claim that every simple element of \( M_{LD} \) is a right divisor of an element \( \Delta_t \), a question we shall address in Chapter IX.

Another consequence of Proposition 5.20 (divisors of \( \Delta \)) is:

Proposition 5.21. (lcm) Simple elements of \( M_{LD} \) are closed under right lcm and operation \( \backslash \).

Proof. Assume that \( a, b \) are simple elements of \( M_{LD} \). Let \( t \) be a term both in the domain of \( LD_a \) and in domain of \( LD_b \). Then \( \Delta_t \) is a common right multiple of \( a \) and \( b \), hence it is a right multiple of \( a \backslash b \). By Proposition 5.20 (divisors of \( \Delta \)), the latter element, which divides an element of the form \( \Delta_t \), is simple. Then \( a \backslash b \) is a right divisor of \( \Delta_t \), so, by Lemma 5.9, it is simple as well.
As an application, we obtain that, if the positive words \( u, v \) can be decomposed into the product of \( p \) and \( q \) simple words respectively, then the determination of the words \( u \backslash v \) and \( v \backslash u \) can be done by computing at most \( pq \) lcm’s of simple words (see Exercise 5.25).

In terms of LD-expansions, we deduce:

**Proposition 5.22.** (simple expansion) The term \( \partial t \) is an LD-expansion of every simple LD-expansion of \( t \)—thus, the simple LD-expansions of a term \( t \) are exactly those LD-expansions of \( t \) of which \( \partial t \) is an LD-expansion.

**Proof.** Assume that \( t' \) is a simple LD-expansion of \( t \). By definition, there is a simple element \( a \) of \( M_{LD} \) satisfying \( t' = t \cdot a \). Now \( a \vee \Delta_t \) is simple. By Proposition VII.3.22 (syntactic confluence), there exists a common right multiple \( b \) of \( a \) and \( \Delta_t \) such that the domain of \( LD_b \) is the intersection of the domains of \( LD_a \) and \( LD_{\Delta_t} \), hence \( t \) belongs to the domain of \( LD_b \), and, therefore, of \( LD_a \vee LD_{\Delta_t} \). Then Corollary 5.19 implies \( a \vee \Delta_t = \Delta_t \), i.e., \( a \) is a left divisor of \( \Delta_t \), and, therefore, \( \partial t \) is an LD-expansion of \( t \cdot a \). ■

Another application is existence of a normal form for \( M_{LD} \).

**Proposition 5.23.** (normal form) Every element of \( M_{LD} \) admits a unique expression of the form \( u_1 \ldots u_p \), where \( u_1, \ldots, u_p \) are permutation-like words and, for \( k \geq 2 \) and \( \alpha \in A \), if \( g_\alpha^+ \) is a left divisor of \( u_k \) in \( M_{LD} \), then \( u_k g_\alpha^+ \) is not simple.

**Proof.** By Proposition II.3.4 (lcm monoid), \( M_{LD} \) is an atomic right lcm monoid. For every address \( \alpha \), the generator \( g_\alpha^+ \) is simple. By Proposition 5.21 (lcm), simple elements make a saturated family in \( M_{LD} \). We then apply Proposition II.3.11 (decomposition). ■

**Exercise 5.24.** (projection) (i) Show that the projection of a permutation-like word on \( A \) is a permutation braid word, and that the projection of a simple word on \( A \) is a positive braid word \( u \) with the property that, in the braid diagram associated with \( u \), any two strands cross at most once.

(ii) Prove that, for every term \( t \), the projection of \( \Delta_t \) is \( \Delta_n \), where \( n \) is the right height of \( t \). Compare their properties.
Exercise 5.25. (simple complement) Let \( A^* \) denote the set of all permutation-like words on \( A \). We say that a word on \( A \) is simple if it represents a simple element of \( M_{LD} \). Our aim is to directly construct a complement \( f^* \) on \( A^* \) as was done in Section I.5 for braids.

(i) Assume that \( u, v \) are simple words on \( A \). Prove that there exist two integers \( p', q' \) such that the words \( u \cdot \phi(q') \) and \( v \cdot \phi(p') \) are simple and \( \phi \) has the same exponent in both; in addition, every term that lies both in the domain of \( LD_u \) and of \( LD_v \) lies in the domain of \( LD_u \cdot LD_{\phi(q')} \) and \( LD_v \cdot LD_{\phi(p')} \). [Hint: For \( w \) a simple word and \( m \) an integer, let \( \hat{\omega}(m) = m + e(1^m, w) \); say that \( r \) is accessible for \( w \) if there exists \( q \) satisfying \( r = \hat{\omega}(q) \) and \( \hat{\omega}(m) < \hat{\omega}(q) \) for \( 0 \leq m < q \); by Lemma 5.14, the accessible integers are the possible exponents of \( \hat{\phi} \) in those simple words one can obtain from \( w \) by multiplying it on the right by some word of the form \( \hat{\phi}^t \). Observe that \( \hat{\omega} \) eventually coincides with the identity, so an accessible \( r \) exists. Let \( r \) be the least accessible, and let \( p', q' \) be the smallest integers satisfying \( r = \hat{\omega}(q') = \hat{\omega}(p') \). Show that at least one of \( p' < r, q' < r \) holds. Assume \( \hat{\omega}(q') < r \); by definition, the exponent of \( 1^q \) in \( u \) is the positive integer \( r - q' \). Let \( t \) be any term in the domain of \( LD_u \). Then \( t \cdot (1^q)^{(r-q')} \) exists, i.e., the right height of \( t \) is at most \( q' + (r - q') + 1 \), i.e., \( r + 1 \). Conclude that \( t \) lies in the domain of \( LD_{\phi(q')} \), and, a fortiori, both in the domains of \( LD_{\phi(q')} \) and of \( LD_{\hat{\phi}(p')} \), hence in the domains of \( LD_u \cdot LD_{\phi(q')} \) and \( LD_v \cdot LD_{\hat{\phi}(p')} \).] Apply the previous construction to \( u = \phi \cdot 11(2) \) and \( v = 1(3) \cdot 11 \). [One finds \( u \cdot \phi(2) \equiv + \phi(4) \cdot 1 \cdot 0, \) and \( v \cdot \phi \equiv + \phi(4) \cdot 11. \)]

(ii) Prove that there exists an explicit mapping \( f^* \) of \( A^* \times A^* \) into \( A^* \) such that, for all permutation-like words \( u, v \), we have \( u \cdot f^*(v, u) \equiv + v \cdot f^*(u, v) \); in addition, the latter words are simple, and the domain of the associated operator is the intersection of the domains of \( LD_u \) and \( LD_v \). [Hint: Use (i) and an induction on the size of \( t^u_{u,v} \).] Apply the construction to the words \( u, v \) of (ii). [One finds \( f^*(v, u) = \phi \cdot 11 \cdot 10 \cdot 0, u \cdot f^*(v, u) = \phi \cdot 1 \cdot 11 \cdot 10 \cdot 0, u = \phi \cdot 1 \cdot 0 \cdot 0 \).]

(iii) Using (ii), re-prove that the simple elements of \( M_{LD} \) are closed under right lcm and operation \( \backslash \), and deduce a new proof of the confluence property.

Exercise 5.26. (degree 2) Say that the term \( t' \) is a degree 2 expansion of the term \( t \) if \( \partial^2 t \) is an LD-expansion of \( t' \). Prove that a degree 2 LD-expansion of \( t \) need not be a simple LD-expansion of a simple LD-expansion of \( t \). [Hint: Consider \( t = x^{[4]} \) and \( t' = t \cdot u \), with \( u = 1 \cdot 1 \cdot \phi \cdot 0 \cdot 00 \cdot 00 \), and prove that \( \partial t'' \) is an LD-expansion of \( t' \) for no \( t'' \) of which \( t' \) is an LD-expansion.] Compare with the case of braids.
Section VIII.6: Notes

[All divisors of $\Delta_n^2$ can be expressed as the product of two simple braids.]

Exercise 5.27. (operator $\partial$) (i) For $u$ a simple word on $A$ define $f(u)$ to be $u \backslash \Delta_t^2$. Prove that $f$ induces a well defined mapping on $M_{LD}$, and that $LD_{f(u)}$ maps $t_u^n$ to $\partial t_u^n$.

(ii) Define $\partial u$ to be $f(u) \backslash \Delta_t^2$. Prove that $\partial$ induces a well defined mapping on $M_{LD}$, and that $LD_{\partial u}$ maps $\partial t_u^n$ to $\partial t_u^n$. Deduce that, if $t'$ is a simple LD-expansion of the term $t$, then $\partial t'$ is a simple LD-expansion of $\partial t$. What is the counterpart of $\partial$ on $B_n^+$? [Answer: The flip automorphism.]

Exercise 5.28. (no right normal form) Let $a = g_1^+ g_0^+ g_1^+ g_0^+$. Prove that $a$ admits no unique maximal simple right divisor in $M_{LD}$. [Hint: Show that $g_1^+ g_0^+ g_1^+$ and $g_0^+ g_1^+$ are maximal simple right divisors of $a$.]

6. Notes

The results of Sections 1, 2, some of the results in Sections 3 and 4, and Exercise 5.25 (simple complement) appear, sometimes implicitly, in (Dehornoy 94a). The results about charged braids are from (Dehornoy 94c). The results about the linear order on $G_{LD}$ and about simple elements have not yet appeared in print.

We think that considering the group $G_{LD}$ gives a really deep insight into the left self-distributivity identity. Although probably more complicated as it requires verifying the coherence and the convergence of the complement involved in $M_{LD}$ first, the syntactic proof of the acyclicity of left division in free LD-systems given in Section 3 may be preferred to the proof given in Chapter I, because it resorts to geometrical properties of left self-distributivity exclusively, rather than to the specific properties of a particular LD-system. Using this proof and the argument of Proposition III.2.18 ($\sigma_i$-positive), we obtain a complete proof of the acyclicity of division in $(B_\infty, \wedge)$, of existence of the braid order, and of decidability of the word problem of Identity (LD): this is the original scheme of (Dehornoy 94a)—and we shall mention in the appendix to Chapter IX that this scheme is the only one working in the case of other identities.

One of the main questions left open is whether $LD_u = LD'_u$ implies $u \equiv^+ u'$ when $u, u'$ are words on $A$. We shall discuss the problem in the next chapter. Let us mention here that the methods of Section 2 do not apply directly as the
blueprints are not positive words in general. Now, we can associate with every term $t$ in $T_1$ positive words, namely the numerator $\chi_t^+$ and the denominator $\chi_t^-$ of $\chi_t$. Experiments suggest that $t' = t \cdot u$ with $u$ a positive word could imply $\chi_{t'}^+ \equiv^+ \chi_t^+ \cdot v$ and $0u \cdot \chi_{t'}^- \equiv^+ \chi_t^- \cdot v$ for some positive $v$, but, even if this is true, it is not clear that these formulas determine $u$ up to $\equiv^+$-equivalence.

No upper bound about the complexity of word reversing in $A^*$ is known, except the coarse bound of Proposition 1.11 (convergence). In the case of $B_\infty$, we have seen that replacing the atomic complement $f_a$ with the simple complement $f_s$ improves the complexity bound, which decreases from exponential to polynomial.

**Question 6.1.** Which upper bound for the complexity of the word problem in $M_{LD}$ can be deduced from using the simple complement $f_s$ of Exercise 5.25?

The injectivity of the mapping $\chi$ on $T_1$ implies that, if $a$ is a special element of $G_{LD}$, then there exists exactly one decomposition $a = b^c$ with $b, c$ special.

**Question 6.2.** Assume that $a$ is an element of $G_{LD}$ that can be expressed as $b^c$; Are $b$ and $c$ unique?

About order properties on terms and on $G_{LD}$, the existence of a possible connection between the braid order and the linear order $<$ on $G_{LD}$ is open.

Finally, we conjecture a positive answer to the following question, which could be connected to Laver’s conjectures of Chapter IV and V:

**Question 6.3.** Is the LD-equivalence class of every term well ordered by $<_L$?