Coxeter-like groups for groups of set-theoretic solutions of the Yang–Baxter equation
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• 1. Set-theoretic solutions of YBE, biracks and RC-quasigroups
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2. YBE-groups and monoids
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• 1. Set-theoretic solutions of YBE, biracks and RC-quasigroups
• 2. YBE-groups and monoids
• 3. Garside germs and Coxeter-like groups
• Original Yang–Baxter Equation:
• **Original Yang–Baxter Equation**: For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$, 
The Yang–Baxter equation

- Original Yang–Baxter Equation: For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \rightarrow V \otimes V$,

$$R_{12}(a) R_{23}(a + b) R_{23}(b) = R_{23}(b) R_{23}(a + b) R_{12}(a).$$

(*)
• **Original Yang–Baxter Equation:** For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$,

\[ R_{12}(a) R_{23}(a + b) R_{23}(b) = R_{23}(b) R_{23}(a + b) R_{12}(a). \] (*)

• Substituting $R$ with $PR$, where $P(x \otimes y) := y \otimes x,$
- **Original Yang–Baxter Equation**: For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \rightarrow V \otimes V$, 
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\] (*)

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\]
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• Substituting $R$ with $PR$, where $P(x \otimes y) := y \otimes x$, (*) becomes

$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}$.

• **Fact:** If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$. 
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• **Fact**: If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$.

• **Definition** (Drinfel’d):
• **Original Yang–Baxter Equation:** For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$, 
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R_{12}(a) R_{23}(a + b) R_{23}(b) = R_{23}(b) R_{23}(a + b) R_{12}(a). \quad (\ast)
\]

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\]

• **Fact:** If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$.

• **Definition** (Drinfel’d): A set-theoretic solution of YBE
The Yang–Baxter equation

• Original Yang–Baxter Equation: For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \rightarrow V \otimes V$,

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• Substituting $R$ with $PR$, where $P(x \otimes y) := y \otimes x$, $(*)$ becomes

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.$$

• Fact: If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$.

• Definition (Drinfel’d): A set-theoretic solution of YBE is a pair $(X, \rho)$, where $X$ is a set and $\rho : X \times X \rightarrow X \times X$ satisfies

$$\rho_{12} \rho_{23} \rho_{12} = \rho_{23} \rho_{12} \rho_{23}.$$
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Fact: If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$.

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Called involutive if $\rho^2 = \text{id}$,
The Yang–Baxter equation

- **Original Yang–Baxter Equation:** For \( V \) a \( \mathbb{C} \)-vector space and \( R : V \otimes V \to V \otimes V \),
  \[
  R_{12}(a) R_{23}(a + b) R_{23}(b) = R_{23}(b) R_{23}(a + b) R_{12}(a). \tag{*}
  \]

- Substituting \( R \) with \( PR \), where \( P(x \otimes y) := y \otimes x \), \( (*) \) becomes
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  R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.
  \]

- **Fact:** If there exists a basis \( X \) of \( V \) s.t. \( R \) preserves \( X \otimes X \) (a very special case!), then \( R \) is determined by the restriction of \( R \) to \( X \otimes X \).

- **Definition (Drinfel’d):** A set-theoretic solution of YBE is a pair \((X, \rho)\), where \( X \) is a set and \( \rho : X \times X \to X \times X \) satisfies
  \[
  \rho_{12} \rho_{23} \rho_{12} = \rho_{23} \rho_{12} \rho_{23}.
  \]
  Called **involutive** if \( \rho^2 = \text{id} \), and **nondegenerate** if, writing \( \rho = (\rho_1, \rho_2) \),
  \[
  \forall s (y \mapsto \rho_1(s, y) \text{ is bijective}),
  \]
• **Original Yang–Baxter Equation:** For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$,
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R_{12}(a) R_{23}(a+b) R_{23}(b) = R_{23}(b) R_{23}(a+b) R_{12}(a).
\] (*)

• Substituting $R$ with $PR$, where $P(x \otimes y) := y \otimes x$, (*) becomes
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• **Fact:** If there exists a basis $X$ of $V$ s.t. $R$ preserves $X \otimes X$ (a very special case!), then $R$ is determined by the restriction of $R$ to $X \otimes X$.

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Called **involutive** if $\rho^2 = \text{id}$, and **nondegenerate** if, writing $\rho = (\rho_1, \rho_2)$,
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\forall s \ (y \mapsto \rho_1(s, y) \text{ is bijective}), \quad \text{and} \quad \forall t \ (x \mapsto \rho_2(x, t) \text{ is bijective}).
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• **Original Yang–Baxter Equation**: For $V$ a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$,

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\((*)\)

• Substituting $R$ with $PR$, where $P(x \otimes y) := y \otimes x$, $(*$) becomes

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R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.
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\forall s \ (y \mapsto \rho_1(s, y) \text{ is bijective}), \quad \text{and} \quad \forall t \ (x \mapsto \rho_2(x, t) \text{ is bijective}).
\]

• Even for $X$ finite, very poorly understood.
• **Definition** (Fenn–Rourke?):
• Definition (Fenn–Rourke?): A birack is a triple \((X, [, ])\) where \([, ]\) are binary operations on \(X\) satisfying

\[
\begin{align*}
(a \cdot b) \cdot ((a \cdot b) \cdot c) &= a \cdot (b \cdot c), \\
(a \cdot b) \cdot ((a \cdot b) \cdot c) &= (a \cdot (b \cdot c)) \cdot (b \cdot c), \\
(a \cdot b) \cdot c &= (a \cdot (b \cdot c)) \cdot (b \cdot c),
\end{align*}
\]
Definition (Fenn–Rourke?): A **birack** is a triple \((X, [\cdot], [\cdot])\) where \([\cdot], [\cdot] \) are binary operations on \(X\) satisfying

\[
\begin{align*}
(a \cdot b) \cdot ((a \cdot b) \cdot c) &= a \cdot (b \cdot c), \\
(a \cdot b) \cdot ((a \cdot b) \cdot c) &= (a \cdot (b \cdot c)) \cdot (b \cdot c), \\
(a \cdot b) \cdot c &= (a \cdot (b \cdot c)) \cdot (b \cdot c),
\end{align*}
\]

and the left-translations of \([\cdot]\) and the right-translations of \([\cdot]\) are one-to-one.
Definition (Fenn–Rourke?): A birack is a triple \((X, \lceil, \rceil)\) where \(\lceil, \rceil\) are binary operations on \(X\) satisfying

\[
\begin{align*}
(a \lceil b) \lceil ((a \lceil b) \rceil c) &= a \lceil (b \rceil c), \\
(a \lceil b) \rceil ((a \lceil b) \rceil c) &= (a \rceil (b \rceil c)) \rceil (b \rceil c), \\
(a \lceil b) \rceil c &= (a \rceil (b \rceil c)) \rceil (b \rceil c),
\end{align*}
\]

and the left-translations of \(\lceil\) and the right-translations of \(\rceil\) are one-to-one. A birack is involutive if, moreover,

\[
\begin{align*}
(a \lceil b) \lceil (a \lceil b) &= a \quad \text{and} \quad (a \lceil b) \rceil (a \lceil b) = b.
\end{align*}
\]
• **Definition** (Fenn–Rourke?): A birack is a triple \((X, [\cdot], [\cdot])\) where \([\cdot], [\cdot]\) are binary operations on \(X\) satisfying

\[
(a \cdot b) \cdot ((a \cdot b) \cdot c) = a \cdot (b \cdot c),
\]

\[
(a \cdot b) \cdot ((a \cdot b) \cdot c) = (a \cdot (b \cdot c)) \cdot (b \cdot c),
\]

\[
(a \cdot b) \cdot c = (a \cdot (b \cdot c)) \cdot (b \cdot c),
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and the left-translations of \([\cdot]\) and the right-translations of \([\cdot]\) are one-to-one. A birack is **involutive** if, moreover,

\[
(a \cdot b) \cdot (a \cdot b) = a \quad \text{and} \quad (a \cdot b) \cdot (a \cdot b) = b.
\]

• **Proposition:** Invol. nondeg. set-theoretic solution YBE \(\iff\) Involutive biracks.
• **Definition (Fenn–Rourke?):** A birack is a triple \((X, [, ])\) where \([, ]\) and \([, ]\) are binary operations on \(X\) satisfying

\[
(a [ b]) \begin{array}{c} [ c] \\ [ c]
\end{array} = a [ (b [ c])],
\]

\[
((a [ b]) [ c]) = (a [ (b [ c])]) [ (b [ c])],
\]

\[
(a [ b]) [ c] = (a [ (b [ c])]) [ (b [ c])],
\]

and the left-translations of \([, ]\) and the right-translations of \([, ]\) are one-to-one. A birack is **involutive** if, moreover,

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(a [ b]) \begin{array}{c} [ c] \\ [ c]
\end{array} = a \quad \text{and} \quad (a [ b]) \begin{array}{c} [ c] \\ [ c]
\end{array} = b.
\]

• **Proposition:** Invol. nondeg. set-theoretic solution YBE \(\iff\) Involutive biracks.

• Proof: Put \(a [ b] := \rho_1(a, b), \quad a \begin{array}{c} [ c] \\ [ c]
\end{array} := \rho_2(a, b),\)
• **Definition** (Fenn–Rourke?): A **birack** is a triple $(X, \lceil, \rceil)$ where $\lceil, \rceil$ are binary operations on $X$ satisfying

\[
(a \lceil b) \lceil ((a \lceil b) \rceil c) = a \lceil (b \rceil c),
\]
\[
(a \lceil b) \lceil ((a \lceil b) \rceil c) = (a \lceil (b \rceil c)) \lceil (b \rceil c),
\]
\[
(a \lceil b) \lceil c = (a \lceil (b \rceil c)) \lceil (b \rceil c),
\]

and the left-translations of $\lceil$ and the right-translations of $\rceil$ are one-to-one. A birack is **involutive** if, moreover,

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(a \lceil b) \lceil (a \lceil b) = a \quad \text{and} \quad (a \lceil b) \lceil (a \lceil b) = b.
\]

• **Proposition**: Invol. nondeg. set-theoretic solution YBE $\iff$ Involutive biracks.

• **Proof**: Put $a \lceil b := \rho_1(a, b)$, $a \lceil b := \rho_2(a, b)$, and use $(X, \lceil, \rceil)$ for colouring braids:
• **Definition** (Fenn–Rourke?): A birack is a triple $(X, \lceil, \rceil)$ where $\lceil, \rceil$ are binary operations on $X$ satisfying

\[
(a \lceil b) \lceil ((a \lceil b) \rceil c) = a \lceil (b \rceil c),
\]
\[
(a \lceil b) \rceil ((a \lceil b) \rceil c) = (a \rceil (b \rceil c)) \rceil (b \rceil c),
\]
\[
(a \lceil b) \rceil c = (a \rceil (b \rceil c)) \rceil (b \rceil c),
\]

and the left-translations of $\lceil$ and the right-translations of $\rceil$ are one-to-one. A birack is **involutive** if, moreover,

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(a \lceil b) \lceil (a \lceil b) = a \quad \text{and} \quad (a \lceil b) \rceil (a \lceil b) = b.
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- **Proposition**: Invol. nondeg. set-theoretic solution YBE $\iff$ Involutive biracks.

- Proof: Put $a \lceil b := \rho_1(a, b)$, $a \rceil b := \rho_2(a, b)$, and use $(X, \lceil, \rceil)$ for colouring braids:
• **Definition** (Fenn–Rourke?): A birack is a triple \((X, \cdot, \cdot)\) where \(\cdot, \cdot\) are binary operations on \(X\) satisfying

\[
(a \cdot b) \cdot ((a \cdot b) \cdot c) = a \cdot (b \cdot c),
\]

\[
(a \cdot b) \cdot ((a \cdot b) \cdot c) = (a \cdot (b \cdot c)) \cdot (b \cdot c),
\]

\[
(a \cdot b) \cdot c = (a \cdot (b \cdot c)) \cdot (b \cdot c),
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and the left-translations of \(\cdot\) and the right-translations of \(\cdot\) are one-to-one. A birack is **involutive** if, moreover,

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(a \cdot b) \cdot (a \cdot b) = a \quad \text{and} \quad (a \cdot b) \cdot (a \cdot b) = b.
\]

• **Proposition**: Invol. nondeg. set-theoretic solution YBE \(\iff\) Involutive biracks.

• Proof: Put \(a \cdot b := \rho_1(a, b),\ a \cdot b := \rho_2(a, b),\) and use \((X, \cdot, \cdot)\) for colouring braids:
• **Definition** (Fenn–Rourke?): A birack is a triple \((X, \lceil, \rceil)\) where \(\lceil, \rceil\) are binary operations on \(X\) satisfying

\[
(a \lceil b) (a \lceil b \rceil c) = a \lceil (b \rceil c),
\]
\[
(a \lceil b) \lceil ((a \lceil b) \rceil c) = (a \lceil (b \rceil c)) \rceil (b \rceil c),
\]
\[
(a \lceil b) ceil c = (a \lceil (b \rceil c)) \rceil (b \rceil c),
\]
and the left-translations of \(\lceil\) and the right-translations of \(\rceil\) are one-to-one. A birack is **involutive** if, moreover,

\[
(a \lceil b) \lceil (a \lceil b) = a \quad \text{and} \quad (a \lceil b) \rceil (a \lceil b) = b.
\]

• **Proposition:** Invol. nondeg. set-theoretic solution YBE \(\iff\) Involutive biracks.

• Proof: Put \(a \lceil b := \rho_1(a, b), a [\lceil b := \rho_2(a, b)\), and use \((X, \lceil, \rceil)\) for colouring braids:
• **Definition (Fenn–Rourke?):** A birack is a triple \((X, \cdot, \circ)\) where \(\cdot, \circ\) are binary operations on \(X\) satisfying

\[
(a \cdot b) \cdot ((a \cdot b) \circ c) = a \cdot (b \circ c),
\]
\[
(a \cdot b) \circ ((a \cdot b) \circ c) = (a \circ (b \circ c)) \cdot (b \circ c),
\]
\[
(a \cdot b) \cdot c = (a \circ (b \circ c)) \cdot (b \circ c),
\]

and the left-translations of \(\cdot\) and the right-translations of \(\circ\) are one-to-one. A birack is **involutive** if, moreover,

\[
(a \cdot b) \cdot (a \circ b) = a \quad \text{and} \quad (a \cdot b) \circ (a \circ b) = b.
\]

• **Proposition:** Invol. nondeg. set-theoretic solution YBE \(\iff\) Involutive biracks.

• **Proof:** Put \(a \cdot b := \rho_1(a, b)\), \(a \circ b := \rho_2(a, b)\), and use \((X, \cdot, \circ)\) for colouring braids:
• **Definition (Rump):**
Definition (Rump): An **RC-system** is a pair \((X, *)\) where * is a binary operation on \(X\) satisfying
\[
(x * y) * (x * z) = (y * x) * (y * z).
\]
Definition (Rump): An **RC-system** is a pair \((X, \ast)\) where \(\ast\) is a binary operation on \(X\) satisfying
\[(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).\]

An **RC-quasigroup** is an RC-system whose left-translations are bijective.
• **Definition (Rump):** An **RC-system** is a pair \((X, \ast)\) where \(\ast\) is a binary operation on \(X\) satisfying
\[
(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).
\]

An **RC-quasigroup** is an RC-system whose left-translations are bijective.

An RC-system is **bijective** if \((s, t) \mapsto (s \ast t, t \ast s)\) is bijective.
• **Definition** (Rump): An **RC-system** is a pair \((X, \ast)\) where \(\ast\) is a binary operation on \(X\) satisfying
\[
(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).
\]
An **RC-quasigroup** is an RC-system whose left-translations are bijective.
An RC-system is **bijective** if \((s, t) \mapsto (s \ast t, t \ast s)\) is bijective.

• **Proposition** (Rump): Involutive biracks \(\iff\) Bijective RC-quasigroups.
• **Definition** (Rump): An **RC-system** is a pair $(X, \ast)$ where $\ast$ is a binary operation on $X$ satisfying
\[(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).\]

An **RC-quasigroup** is an RC-system whose left-translations are bijective.
An RC-system is **bijective** if $(s, t) \mapsto (s \ast t, t \ast s)$ is bijective.

• **Proposition** (Rump): Involutional biracks $\iff$ Bijective RC-quasigroups.

• Proof: For $(X, [, ]) \text{ an involutive birack, put } a \ast b := \text{unique } c \text{ satisfying } a [, ] b = c.$
• **Definition (Rump):** An **RC-system** is a pair \((X, \ast)\) where \(\ast\) is a binary operation on \(X\) satisfying
\[
(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).
\]
An **RC-quasigroup** is an RC-system whose left-translations are bijective.
An RC-system is **bijective** if \((s, t) \mapsto (s \ast t, t \ast s)\) is bijective.

• **Proposition (Rump):** Involutive biracks \(\iff\) Bijective RC-quasigroups.

• **Proof:** For \((X, [,])\) an involutive birack, put \(a \ast b := \) unique \(c\) satisfying \(a ] b = c\).
For \((X, \ast)\) a bijective RC-system, put \(a ] b := \) the unique \(c\) satisfying \(a \ast b = c\).
• **Definition** (Rump): An **RC-system** is a pair \((X, \ast)\) where \(\ast\) is a binary operation on \(X\) satisfying
\[
(x \ast y) \ast (x \ast z) = (y \ast x) \ast (y \ast z).
\]
An **RC-quasigroup** is an RC-system whose left-translations are bijective. An RC-system is **bijective** if \((s, t) \mapsto (s \ast t, t \ast s)\) is bijective.

• **Proposition** (Rump): Involutive biracks \(\iff\) Bijective RC-quasigroups.

• **Proof:** For \((X, \lceil, \rceil)\) an involutive birack, put \(a \ast b := \text{unique } c \text{ satisfying } a \lceil b = c\). For \((X, \ast)\) a bijective RC-system, put \(a \lceil b := \text{the unique } c \text{ satisfying } a \ast b = c\). 

\[\begin{array}{c}
r = a \lceil (b \rceil c) = (a \lceil b) \rceil ((a \lceil b) \rceil c) \\
\hline
r = a \lceil b \\
\hline
s = a \lceil b \\
\hline
r \ast t = (a \lceil b) \rceil c \\
\hline
r \ast t = (a \lceil b) \rceil c \\
\hline
s \ast t = a \lceil b \\
\hline
s \ast r = (a \lceil b) \rceil c \\
\hline
(s \ast r) \ast (s \ast t) = (a \lceil b) \rceil (a \lceil b) \rceil (a \lceil b) \rceil c \\
\hline
(s \ast r) \ast (s \ast t) = (a \lceil b) \rceil (a \lceil b) \rceil (a \lceil b) \rceil c \\
\hline
(s \ast r) \ast (s \ast t) = (a \lceil b) \rceil (a \lceil b) \rceil (a \lceil b) \rceil c \\
\end{array}\]
• **Definition** (Etingof–Schedler–Soloviev): For \((X, \rho)\) an invol. nondeg. set-theoretic solution of YBE, the **structure group** of \((X, \rho)\) is

\[
G := \langle X \mid \{ab = a'b' \mid (a', b') = \rho(a, b)\} \rangle.
\]
• **Definition** (Etingof–Schedler–Soloviev): For \((X, \rho)\) an invol. nondeg. set-theoretic solution of YBE, the **structure group** of \((X, \rho)\) is

\[
G := \langle X \mid \{ab = a'b' \mid (a', b') = \rho(a, b)\}\rangle.
\]

• Equivalently: For \((X, \ast)\) a bijective RC-quasigroup, the **structure group** of \((X, \ast)\) is

\[
G := \langle X \mid \{s(s \ast t) = t(t \ast s) \mid s, t \in X\}\rangle.
\]
• **Definition** (Etingof–Schedler–Soloviev): For \((X, \rho)\) an invol. nondeg. set-theoretic solution of YBE, the **structure group** of \((X, \rho)\) is

\[ G := \langle X \mid \{ab = a'b' \mid (a', b') = \rho(a, b)\} \rangle. \]

• Equivalently: For \((X, \ast)\) a bijective RC-quasigroup, the **structure group** of \((X, \ast)\) is

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• **Main (open) question**: Investigate “YBE-groups” and “YBE-monoids” from an algebraic and geometric viewpoint.
**Fact:** The Cayley graph of an YBE-group (resp. monoid) with $n$ atoms resembles that of $\mathbb{Z}^n$ (resp. $\mathbb{N}^n$).
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![Diagram of the Cayley graph for the example given in the text.](image)

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• **Theorem** (Gateva–Van den Bergh, Jespers–Okniński): YBE-monoids with atom set $X$
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YBE-monoids with atom set \( X \) \iff Monoids with an \( X \)-based \( I \)-structure.
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\[ \uparrow \]

\[ \mathcal{G}_n \]

The whole structure of \( B_n^+ \) (and \( B_n \)) is encoded

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+ a natural (set-theoretic) section \( \sigma : f \mapsto f \) from \( \mathbb{G}_n \) to \( B_n^+ \) s.t. \( \mathbb{G}_n \) is a germ for \( B_n^+ \).

\[ \langle \mathbb{G}_n \mid \{fg = h \mid \ell(f) + \ell(g) = \ell(h)\} \rangle^+ = B_n^+ \]

length of a permutation = number of inversions

- The whole structure of \( B_n^+ \) (and \( B_n \)) is encoded
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• Proof: Combines the I-structure and the Garside structure;
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• **Proof**: Combines the I-structure and the Garside structure; key point: “RC-calculus”.
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• **Remark**: Special case of class 2 previously addressed by Chouraqui and Godelle.
• **Example:** Again $X = \{a, b, c\}$ with $x * y = f(y)$ and $f : a \mapsto b \mapsto c \mapsto a$. 
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• **Question:** Which finite groups arise?
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• **Question:** Which finite groups arise? What are their linear representations?
  (known: for $\#X = n$, there exists an $n$-dimensional unitary representation)

  e.g., above: $a \mapsto \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}$, $c \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix}$
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